Homework 4

1. For $1 \le i \le n$ let X_i be independent random variables with $\Pr[X_i = 1] = 1/i$ and $\Pr[X_i = 0] = 1 - 1/i$. Set $Y_n = \sum_{i=1}^{n} X_i$. Find precise formulas (as summations; there will not be a closed form) and asymptotic formulas for both $E[Y_n]$ and $\mathrm{Var}(Y_n)$. Use Chebyshev's Inequality to show that for any $\varepsilon > 0$,
$$\lim_{n\to\infty} \Pr\big[|Y_n - E[Y_n]| > \varepsilon E[Y_n]\big] = 0.$$

2. (a) Let $G \sim G(n,p)$ with $p = cn^{-1/2}$. Let v, w be two fixed vertices of G. Let X be the number of paths of length two from v to w. Find the asymptotic values of $E[X]$ and $\mathrm{Var}(X)$ as a function of c for fixed c as $n \to \infty$.

   (b) Let $G \sim G(n,p)$ with $p = cn^{-2/3}$. Let v, w be two fixed vertices of G. Let X be the number of paths of length three from v to w. Find the asymptotic values of $E[X]$ and $\mathrm{Var}(X)$ as a function of c for fixed c as $n \to \infty$.

3. Consider a drawing of a graph G in the plane with v vertices, e edges, and κ crossings (a crossing is a pair of edges with distinct endpoints that cross). In this exercise we find a lower bound for κ as a function of v and e.

   (a) Show that if $\kappa = 0$ then $e \le 3v$ (and in fact $e \le 3v - 6$ if $v \ge 3$).

   (b) Show that $\kappa \ge e - 3v$ (and in fact $\kappa \ge e - (3v - 6)$ if $v \ge 3$).

   (c) Take a random subset U of the vertices, where for each vertex P, $\Pr[P \in U] = p$ and these events are mutually independent. Let X, Y, Z be the expected numbers of vertices, edges, and crossings respectively in the restriction of the drawing to U. Show that $E[X] = vp$, $E[Y] = ep^2$, and $E[Z] = \kappa p^4$.

   (d) From parts (b) and (c) give a condition on κ of the form: For all $p \in [0,1]$, yadda yadda yadda.

   (e) Use calculus to derive a theorem in the form: If e is at least blip blip blip then κ is at least blah blah blah. The blip blip blip condition will simply reflect that p must lie in $[0,1]$.

4. Let A be a random $n \times n$ matrix of zeroes and ones where $\Pr[A_{ij} = 1] = p = cn^{-1/2}$ and the a_{ij} are independent. We consider asymptotics for fixed $c > 0$ and $n \to \infty$.

   (a) Find the asymptotic probability that $a_{11} = 1$ and there is no 3×3 all-ones submatrix that includes a_{11}. (Use Janson's Inequality, first conditioning on $a_{11} = 1$.)

   (b) Revisit Problem 5 of Homework 1, asking for the maximal number $f(n)$ of ones in a ninefree $n \times n$ matrix. Use the above result to get a new asymptotic lower bound on $f(n)$ with a different constant.

5. The van der Waerden function $W(k)$ is the least n so that for any 2-coloring of $1, \ldots, n$ there is a monochromatic arithmetic progression $a, a+d, a+2d, \ldots, a+(k-1)d$ with k terms. The existence of $W(k)$ is known as van der Waerden's Theorem. Here we look at lower bounds.

   Thus $W(k) > n$ means there exists a coloring of $1, \ldots, n$ without a monochromatic arithmetic progression with k terms. Consider a random coloring and for each $S = \{a, a+d, a+2d, \ldots, a+(k-1)d\}$ let B_S be the event that S is monochromatic.

   (a) Use standard probabilistic methods to derive a lower bound on $W(k)$. Find the asymptotics of the lower bound.

   (b) Use the Lovász Local Lemma to derive a lower bound on $W(k)$. Find the asymptotics of the lower bound.