A trio of research projects with undergraduates

Enrique Treviño

LAKE FOREST COLLEGE

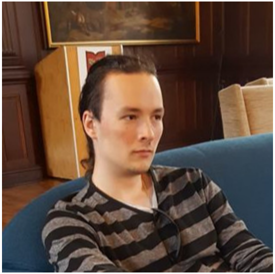
Seminario Coloquio GRACIA-Red Matemática
April 1, 2022

The trio of projects

1. *On generalizing happy numbers to fractional base number systems* with **Mikita Zhylinski**, Lake Forest College.
2. *On a sequence related to the factoradic representation of an integer* with **Maximiliano Sánchez Garza**, Universidad Autónoma de Nuevo León.
3. *Generalizing Parking Functions with Randomness* with **Melanie Tian**, Lake Forest College.

Project 1

On generalizing happy numbers to fractional base number systems with **Mikita Zhylinski**, Lake Forest College.



Happy numbers

- Let $S(n)$ be the sum of the squares of the digits of n.
- Consider iterating S on positive integers.
- The number n, after enough iterations of S, eventually reaches 1 or it eventually reaches the cycle

$$4 \to 16 \to 37 \to 58 \to 89 \to 145 \to 42 \to 20 \to 4.$$

- We call n *happy* if n eventually reaches 1 after enough iterations of S.
- 13 is happy since

$$13 \to 10 \to 1.$$

- Happy numbers are sequence A007770 in OEIS.

Proof that iterations of S have two possibilities

- If n has $m \geq 4$ digits, then

$$S(n) \leq 81m < 10^m.$$

- If $n \geq 244$ has 3 digits, then

$$S(n) \leq 243 < n.$$

- Therefore, for $n \geq 244$, $S(n) < n$.
- We need only analyze what cycles are reached by positive integers ≤ 243. This can be checked with a computer.

Some generalizations of happy numbers

- Let $S_{e,b}(n)$ be the sum of the e-th powers of the base b digits of n. For example
$$S_{3,5}(13) = 2^3 + 3^3 = 35.$$
- Grundman and Teeple, in 2001, generalized the notion of happy numbers to e-th power b-happy numbers for numbers that reach 1 after repeated iteration of $S_{e,b}$.
- Grundman and Teeple were able to find the cycles that can be reached for $S_{e,b}$ when $e \in \{2, 3\}$ and $2 \leq b \leq 10$.

TABLE 1. Fixed points and cycles of $S_{2,b}$, $2 \leq b \leq 10$

Base	Fixed Points and Cycles
2	1
3	1, 12, 22 $2 \to 11 \to 2$
4	1
5	1, 23, 33 $4 \to 31 \to 20 \to 4$
6	1 $32 \to 21 \to 5 \to 41 \to 25 \to 45 \to 105 \to 42 \to 32$
7	1, 13, 34, 44, 63 $2 \to 4 \to 22 \to 11 \to 2$ $16 \to 52 \to 41 \to 23 \to 16$
8	1, 24, 64 $4 \to 20 \to 4$ $5 \to 31 \to 12 \to 5$ $15 \to 32 \to 15$
9	1, 45, 55 $58 \to 108 \to 72 \to 58$ $82 \to 75 \to 82$
10	1 $4 \to 16 \to 37 \to 58 \to 89 \to 145 \to 42 \to 20 \to 4$

More generalizations and other results

- El-Sedy and Siksek, in 2000, proved there are arbitrarily long sequences of consecutive happy numbers.
- Grundman and Teeple generalized El-Sedy's and Siksek's result to other bases and to other exponents.
- Grundman and Harris generalized the result to negative bases. (Yes, there is a way to represent numbers allowing for negative bases).
- Bland, Cramer, de Castro, Domini, Edgar, Johnson, Klee, Koblitz, and Sundaresan, in 2017, generalized the concept of happy numbers to **fractional bases** (but not the proof about arbitrarily long sequences of consecutive fractional based happy numbers).

Fractional Base

For any p/q with $\gcd(p,q) = 1$ and $p > q$, for every positive integer n, there exist *fractional digits* $a_0, a_1, \ldots, a_r$ satisfying $0 \leq a_i < p$ for $i \in \{0, 1, \ldots, r-1\}$ and $0 < a_r < p$ such that

$$n = \sum_{i=0}^{r} a_i \left(\frac{p}{q}\right)^i.$$

We will write

$$n = \overline{a_r a_{r-1} a_{r-2} \ldots a_2 a_1 a_0}_{\frac{p}{q}}.$$

n	n in base 3/2	n	n in base 3/2
0	$\overline{0}_{\frac{3}{2}}$	6	$\overline{210}_{\frac{3}{2}}$
1	$\overline{1}_{\frac{3}{2}}$	7	$\overline{211}_{\frac{3}{2}}$
2	$\overline{2}_{\frac{3}{2}}$	8	$\overline{212}_{\frac{3}{2}}$
3	$\overline{20}_{\frac{3}{2}}$	9	$\overline{2100}_{\frac{3}{2}}$
4	$\overline{21}_{\frac{3}{2}}$	10	$\overline{2101}_{\frac{3}{2}}$
5	$\overline{22}_{\frac{3}{2}}$	11	$\overline{2102}_{\frac{3}{2}}$

Table: The first 12 non-negative integers in the 3/2 base number system.

Our research questions

Working with **Mikita Zhylinski**, we worked on the following questions raised by Bland et. al.

1. Can we find the cycles reached by $S_{e,b}$ for different e-th powers when $p/q = 3/2$?
2. Can we find the cycles reached by $S_{e,b}$ for different p/q when we restrict to $e = 2$?
3. Are there positive integers n of arbitrarily large **height**?

Answer to Question 1

e	Cycles	n^*
1	(1), (2)	2
2	(1), (5, 8, 9)	8
3	(1), (9), (10), (17, 18)	32
4	(1), (51), (52)	77
5	(1), (131), (98, 99)	185
6	(1), (197, 260, 387, 323, 263, 450), (324, 131, 259)	419
7	(1), (771, 516, 643, 518)	1211
8	(1), (1539, 775, 1284), (1287, 1794, 1796, 2052), (1032), (1033)	2723
9	(1), (2566), (2565)	6557
10	(1), (10247)	13118
11	(1), (14342, 16388, 14344), (14341), (14340)	27968
12	(1), (28678), (28677)	62933

Table: Cycles reached when iterating $S_{e,\frac{3}{2}}$, and the value of n^* for different values of e.

Answer to Question 2

p/q	$e=2$	$e=3$	$e=4$
5 / 2	(16, 6, 5, 4), (32, 24, 29); $n^* = 39$	(65), (163, 190, 73, 118, 64), (81), (80), (66), (17); $n^* = 239$	(371, 276, 275, 274), (355, 130, 113), (195, 353); $n^* = 1039$
5 / 3	(34, 50), (25), (26), (59), (23), (11), (10); $n^* = 59$	(100, 38, 64, 102, 46), (101, 39), (127, 107, 73, 135), (162), (193), (190, 166, 218), (199, 237); $n^* = 284$	(772, 804, 454, 788, 950, 658, 934, 1126, 1028, 1202, 868, 936, 390), (1027, 1137, 1125), (1122, 994), (1299), (101), (100); $n^* = 1324$
5 / 4	(66, 55), (50), (58, 75, 49, 56, 67), (74, 83), (51); $n^* = 74$	(311, 251, 247, 231, 371), (361), (417), (374), (360), (314), (424, 418, 436, 272, 328, 364); $n^* = 464$	(1786, 1880, 1403, 1594, 1659, 2011, 2075, 1579, 2057, 1947, 1688, 1229, 1641, 1676, 1946, 1673, 1851, 1592, 1419, 1974, 2058, 2012, 2090); $n^* = 2639$
7 / 2	(25, 52), (97); $n^* = 97$	(341, 591, 376, 143, 187, 216, 352, 25, 280, 244, 469, 63, 128, 44, 141, 161, 197, 73, 307, 467, 377, 234, 182, 91), (35), (288, 343, 9, 16, 72), (36), (189), (190), (468); $n^* = 615$	(914, 2065, 1953, 1538, 2819, 2690, 2210, 1507, 1491, 2610, 1856, 1348, 1666, 259, 1808, 2659, 3136, 1824), (1634, 1731, 994), (371, 34, 1313), (130, 354, 289, 1938, 3265, 2930, 1474, 1570), (451, 195, 2177, 1554, 179, 513, 2034, 2530); $n^* = 5417$

Table: Cycles reached when iterating $S_{e,\frac{p}{q}}$, and the value of n^* for different values of e and p/q.

Height

For a number n, the height is the number of iterations of $S_{e,b}$ it takes to reach a cycle.

Examples for $e = 2, b = 10$:

- The height of 13 is 2 since $13 \to 10 \to 1$.
- The height of 14 is 6 because

$$14 \to 17 \to 50 \to 25 \to 29 \to 85 \to 89.$$

- The height of 15 is 4 because

$$15 \to 26 \to 68 \to 100 \to 1.$$

Arbitrary Height

Let's prove that for the happy function $S_{2,10}$, the height of a number can be arbitrarily large.

- 14 has height 6.
- The number

$$m = \underbrace{11\cdots 1}_{14}$$

  has height 7 since $S(m) = 14$.
- Then the number with m 1's has height 8 and so on.

Arbitrary Height for Fractional Bases

Theorem

Let $p > q$ be positive integers with $\gcd(p, q) = 1$, and let e and H be positive integers. If $q = 2$ or $e = 1$, then there exists an integer n such that the height of n is H.

Sketch of the Proof

- We will show that for $k \geq 2^e$, there is an even integer n such that $S_{e,p/2}(n) = k$.
- Taking $n = 2$ we get that it's true for 2^e.
- Assume there is an even m such that $S_{e,p/2}(m) = k$.
- Let $m = 2^b c$ with $b \geq 1$ and c odd. Write m in base $p/2$ as

$$m = \overline{a_r a_{r-1} \cdots a_1 a_0}.$$

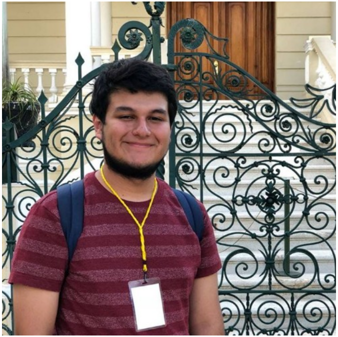
- Then

$$\left(\frac{p}{2}\right)^b m + 1 = \overline{a_r a_{r-1} \cdots a_1 a_0 \underbrace{0 \cdots 0}_{b-1} 1},$$

  where there are $b - 1$ zero digits.
- $(p/2)^b m + 1$ is even. Furthermore, since it has the same digits as before with $b - 1$ zeroes added and one 1 added, the sum of the e-th powers of the digits is $k + 1$.

Project 2

On a sequence related to the factoradic representation of an integer with **Maximiliano Sánchez Garza**, Universidad Autónoma de Nuevo León.



Factoradic Representation

- Every positive integer n can be written uniquely in the form

$$n = \sum_{i=1}^{k} a_i \cdot i!,$$

  for some positive integer k satisfying $1 \le a_k \le k$, and $0 \le a_i \le i$ for $1 \le i \le k-1$.
- We call this the **factoradic expansion** of n.
- We will use the notation $n = (a_k a_{k-1} \cdots a_1)_!$ to express a number written in its factoradic expansion.
- For example, $8 = 110_!$ because $8 = 0 \cdot 1! + 1 \cdot 2! + 1 \cdot 3!$.

Factoradic Happy Numbers

- Carlson, Goedhart, and Harris, in 2020, generalized the concept of happy numbers to factoradic expansions as follows: let $S_{r,!}(n)$ be the sum of the r-th powers of the factoradic digits of a number n, then a positive number n is an r-power factoradic happy number if there exists an integer k such that $S_{r,!}^{k}(n) = 1$ (the k-iteration of $S_{r,!}$ is 1).
- Their main theorem is that for $r \in \{1, 2, 3, 4\}$, there exist arbitrarily long sequences of consecutive r-power factoradic happy numbers.

Motivating Result

Let r be a positive integer and define j_r to be the smallest positive integer n satisfying

$$n! > n^{r-1}.$$

Theorem (Carlson, Goedhart, Harris, 2020)

Let r be a positive integer satisfying $2 \leq r \leq 30$. Write n in its factoradic expansion as $n = \sum_{i=1}^{k} a_i i!$ with $1 \leq a_k \leq k$, and $0 \leq a_i \leq i$ for $i \in \{1, 2, \ldots, k-1\}$. Let

$$S_{r,!}(n) = \sum_{i=1}^{k} a_i^r.$$

Then for $n \geq (j_r + 1)!$,

$$S_{r,!}(n) < n.$$

The sequence j_r

Let r be a positive integer and define j_r to be the smallest positive integer n satisfying

$$n! > n^{r-1}.$$

- The first 20 values of j_r in the On-line Encyclopedia of Integer Sequences are

$$\{2, 3, 4, 6, 7, 8, 10, 11, 12, 14, 15, 16, 18, 19, 20, 22, 23, 24, 25, 27\}.$$

- It is sequence A230319 in OEIS.

Properties of j_r

Some properties of j_r we proved:

- $j_{r+1} - j_r \in \{1, 2\}$ for all r.
- Let $\varepsilon > 0$ be a real number. Then there exists M such that, for integers $r > M$, we have that $j_r < (1 + \varepsilon)r$.
- For a positive integer r, there exists a real number θ_r such that

$$j_r = r + \frac{r}{\log r} + \theta_r \left(\frac{r}{\log r} \right),$$

with $\theta_r \to 0$ as $r \to \infty$.

j-prime

We know $j_{r+1} - j_r \in \{1, 2\}$. Let r be a j-prime if $j_{r+1} - j_r = 2$.

Theorem

Let $J(x) = \{r \leq x \mid r \text{ is a } j\text{-prime}\}$. Then

$$J(x) \sim \frac{x}{\log x} \sim \pi(x).$$

Sums of Powers

For the theorem we want to prove we need to control sums of powers. There's a beautiful classical theorem we will need

Theorem

$$\sum_{i=1}^{n} i^r = \frac{1}{r+1} \sum_{k=0}^{r} \binom{r+1}{k} B_k n^{r+1-k},$$

where B_k are the Bernoulli numbers with the convention that $B_1 = \frac{1}{2}$.

Bernoulli numbers

- For $k > 0$, $B_k < 0$ if and only if $k \equiv 0 \pmod 4$.
- We have
$$\sum_{k=0}^{\infty} \frac{B_k}{k!} = \frac{e}{e-1}. \tag{1}$$
- For $k > 0$, we have
$$B_{2k} = \frac{(-1)^{k+1} 2(2k)!}{(2\pi)^{2k}} \zeta(2k). \tag{2}$$
- $B_{2k+1} = 0$ for $k \geq 1$.

A useful inequality

Lemma

Let $r \geq 2$ be an integer. Then, for an integer $n \geq r+1$,

$$\sum_{i=1}^{n} i^r \leq \frac{Cn^{r+1}}{r+1}$$

where $C = \frac{e}{e-1} + \frac{\pi^4}{45(16\pi^4-1)} = 1.583\ldots$.

Note: The usual "straightforward" inequality is

$$\sum_{i=1}^{n} i^r \leq \int_1^{n+1} t^r \,\mathrm{d}t = \frac{(n+1)^{r+1}-1}{r+1}.$$

Proof of Lemma

For $n \geq r+1$, observe that $\binom{r+1}{k} \leq \frac{(r+1)^k}{k!} \leq \frac{n^k}{k!}$.

$$\sum_{i=1}^{n} i^r \leq \frac{1}{r+1}\left(1 + \sum_{\substack{1\leq k\leq r \\ k \not\equiv 0 \pmod 4}} \frac{n^k}{k!} B_k n^{r+1-k}\right) \leq \frac{n^{r+1}}{r+1}\left(1 + \sum_{\substack{1\leq k\leq r \\ k \not\equiv 0 \pmod 4}} \frac{B_k}{k!}\right).$$

Now, using properties of Bernoulli numbers and that $\zeta(4k) \leq \zeta(4) = \frac{\pi^4}{90}$, we have

$$\begin{aligned}
1 + \sum_{\substack{1\leq k\leq r \\ k \not\equiv 0 \pmod 4}} \frac{B_k}{k!} &\leq \sum_{k=0}^{\infty} \frac{B_k}{k!} - \sum_{k=1}^{\infty} \frac{B_{4k}}{(4k)!} \\
&= \frac{e}{e-1} + 2\sum_{k=1}^{\infty} \frac{1}{(2\pi)^{4k}} \zeta(4k) \\
&\leq \frac{e}{e-1} + 2\zeta(4) \frac{1}{(2\pi)^4 \left(1 - \frac{1}{(2\pi)^4}\right)} \\
&= \frac{e}{e-1} + \frac{\pi^4}{45(16\pi^4 - 1)}.
\end{aligned}$$

Main Theorem

Theorem

Let r be a positive integer. Write n in its factoradic expansion as $n = \sum_{i=1}^{k} a_i i!$ with $1 \leq a_k \leq k$, and $0 \leq a_i \leq i$ for $i \in \{1, 2, \ldots, k-1\}$. Let

$$S_{r,!}(n) = \sum_{i=1}^{k} a_i^r.$$

Then for $n \geq (j_r + 1)!$,

$$S_{r,!}(n) < n.$$

Project 3

Generalizing Parking Functions with Randomness with **Melanie Tian**, Lake Forest College.

Parking Functions

- Consider n cars $C_1, C_2, \ldots, C_n$ that want to park in a parking lot with parking spaces $1, 2, \ldots, n$ that appear in order.
- Each car C_i has a parking preference $\alpha_i \in \{1, 2, \ldots, n\}$.
- The cars appear in order, if their preferred parking spot is not taken, they take it, if the parking spot is taken, they move forward until they find an empty spot. If they don't find an empty spot, they don't park.
- An n-tuple $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ is said to be a parking function, if this list of preferences allows every car to park under this algorithm.
- For example $(2, 1, 1, 2)$ is a parking function while $(4, 3, 3, 1)$ is not.

Counting parking functions

Theorem (Konheim, Weiss, 1966)

Given a positive integer n, The number of parking functions is

$$(n+1)^{n-1}.$$

Proof.

Imagine we add an $n+1$-th parking spot and we wrap-around, so a car can move forward, take the $n+1$-th spot and if even that is taken, they can come back to spot 1.
Under this process, everyone ends up parking and there would be one empty parking spot. An n-tuple is a parking function if and only if the empty parking slot is the $n+1$-th slot. There are $(n+1)^n$ possible tuples and there are $n+1$ possible empty spots, so $(n+1)^{n-1}$ of them are parking functions. □

A variant with randomness

Suppose we change the parking algorithm as follows: If a preferred parking spot is taken, then the car continues forward with probability p and backwards with probability $1 - p$.

- For $p = 1/2$, the tuple $(2, 1, 1, 2)$ has probability 1/4 of having all cars parked.
- For $p = 1/2$, the tuple $(4, 3, 3, 1)$ has probability 1/2 of having all cars parked.

A couple of theorems on this variant

Theorem

The expected number of parking functions is

$$(n+1)^{n-1}.$$

Theorem

An n-tuple $(\alpha_1, \ldots, \alpha_n)$ *has probability 1 of parking if and only if* $\{\alpha_1, \alpha_2, \ldots, \alpha_n\} = \{1, 2, \ldots, n\}$. *Furthermore, if* $\{\alpha_1, \alpha_2, \ldots, \alpha_n\} \neq \{1, 2, \ldots, n\}$, *then the probability of parking is greater than 0.*

Naples parking

Consider the following variant, called Naples-parking:

- If a car is parked in C_i's preferred spot, then C_i will check if the previous spot is taken, if not, he takes that spot, otherwise C_i continues forward.

A generalization is k-Naples parking.

- If a car is parked in C_i's preferred spot, then C_i will back up k spots and move forward until it finds a spot (if a spot is available).

Counting Naples parking functions

Theorem (Christensen, Harris, Jones, Loving, Ramos Rodríguez, Rennie, Rojas Kirby, 2020)

If k, n are nonnegative integers with $k < n$, then the number $N_k(n+1)$ of k-Naples parking functions of length $n+1$ is counted recursively by

$$N_k(n+1) = \sum_{i=0}^{n} \binom{n}{i} \min((i+1)+k, n+1) N_k(i)(n-i+1)^{n-i-1}.$$

Variant introducing randomness

Suppose we consider k-Naples parking, but instead of a car moving the k spaces backward automatically, they decide with probability p to take k spaces back or just stay in the spot.
Some examples with $k = 1$ and $p = 1/2$.

- The tuple $(2, 1, 1, 2)$ has probability 1 of parking.
- The tuple $(4, 3, 3, 1)$ has probability 1/2 of parking.

Probability for all preferences when $n = 4$

Preference	1111	1112	1113	1114	1121	1122	1123	1124
Probability	1	1	1	1	1	1	1	1
Preference	1131	1132	1133	1134	1141	1142	1143	1144
Probability	1	1	1	1	1	1	1	1/2
Preference	1211	1212	1213	1214	1221	1222	1223	1224
Probability	1	1	1	1	1	1	1	1
Preference	1231	1232	1233	1234	1241	1242	1243	1244
Probability	1	1	1	1	1	1	1	1/2
Preference	1311	1312	1313	1314	1321	1322	1323	1324
Probability	1	1	1	1	1	1	1	1
Preference	1331	1332	1333	1334	1341	1342	1343	1344
Probability	1	1	3/4	1/2	1	1	1/2	0
Preference	1411	1412	1413	1414	1421	1422	1423	1424
Probability	1	1	1	1/2	1	1	1	1/2
Preference	1431	1432	1433	1434	1441	1442	1443	1444
Probability	1	1	1/2	0	1/2	1/2	1/4	0
Preference	2111	2112	2113	2114	2121	2122	2123	2124
Probability	1	1	1	1	1	1	1	1
Preference	2131	2132	2133	2134	2141	2142	2143	2144
Probability	1	1	1	1	1	1	1	1/2
Preference	2211	2212	2213	2214	2221	2222	2223	2224
Probability	1	1	1	1	1	7/8	3/4	3/4
Preference	2231	2232	2233	2234	2241	2242	2243	2244
Probability	1	3/4	1/2	1/2	1	3/4	1/2	1/4
Preference	2311	2312	2313	2314	2321	2322	2323	2324
Probability	1	1	1	1	1	3/4	1/2	1/2
Preference	2331	2332	2333	2334	2341	2342	2343	2344
Probability	1	1/2	0	0	1	1/2	0	0
Preference	2411	2412	2413	2414	2421	2422	2423	2424
Probability	1	1	1	1/2	1	3/4	1/2	1/4
Preference	2431	2432	2433	2434	2441	2442	2443	2444
Probability	1	1/2	0	0	1/2	1/4	0	0
Preference	3111	3112	3113	3114	3121	3122	3123	3124
Probability	1	1	1	1	1	1	1	1
Preference	3131	3132	3133	3134	3141	3142	3143	3144
Probability	1	1	3/4	1/2	1	1	1/2	0
Preference	3211	3212	3213	3214	3221	3222	3223	3224
Probability	1	1	1	1	1	3/4	1/2	1/2
Preference	3231	3232	3233	3234	3241	3242	3243	3244
Probability	1	1/2	0	0	1	1/2	0	0
Preference	3311	3312	3313	3314	3321	3322	3323	3324
Probability	1	1	3/4	1/2	1	5/8	1/4	1/4
Preference	3331	3332	3333	3334	3341	3342	3343	3344
Probability	3/4	3/8	0	0	1/2	1/4	0	0
Preference	3411	3412	3413	3414	3421	3422	3423	3424
Probability	1	1	1/2	0	1	1/2	0	0
Preference	3431	3432	3433	3434	3441	3442	3443	3444
Probability	1/2	1/4	0	0	0	0	0	0
Preference	4111	4112	4113	4114	4121	4122	4123	4124
Probability	1	1	1	1/2	1	1	1	1/2
Preference	4131	4132	4133	4134	4141	4142	4143	4144
Probability	1	1	1/2	0	1/2	1/2	1/4	0
Preference	4211	4212	4213	4214	4221	4222	4223	4224
Probability	1	1	1	1/2	1	3/4	1/2	1/4
Preference	4231	4232	4233	4234	4241	4242	4243	4244
Probability	1	1/2	0	0	1/2	1/4	0	0
Preference	4311	4312	4313	4314	4321	4322	4323	4324
Probability	1	1	1/2	0	1	1/2	0	0
Preference	4331	4332	4333	4334	4341	4342	4343	4344
Probability	1/2	1/4	0	0	0	0	0	0

Generalizing the recursion formula

Theorem (CHJLR-RRR-K, 2020)

If k, n are nonnegative integers with $k < n$, then the number $N_k(n)$ of k-Naples parking functions of length n is counted recursively by

$$N_k(n) = \sum_{i=0}^{n-1} \binom{n-1}{i} N_k(i)(n-i)^{n-i-2} \min((i+1)+k, n).$$

Theorem

Let $T_{k,p}(n)$ be the expected number of parking preferences.

$$T_{k,p}(n) = \sum_{i=0}^{n-1} \binom{n-1}{i} T_{k,p}(i)(n-i)^{n-i-2}(i+1+p\min\{k, n-i-1\})$$

Do you see a Pattern?

p	0	1/64	2/64	3/64	4/64	5/64	6/64	7/64
$f(p)$	339472	1	136	1	2194	1	209	1
p	8/64	9/64	10/64	11/64	12/64	13/64	14/64	15/64
$f(p)$	12466	1	140	1	3107	1	143	1
p	16/64	17/64	18/64	19/64	20/64	21/64	22/64	23/64
$f(p)$	40610	1	141	1	1361	1	74	1
p	24/64	25/64	26/64	27/64	28/64	29/64	30/64	31/64
$f(p)$	14253	1	75	1	1589	1	148	1
p	32/64	33/64	34/64	35/64	36/64	37/64	38/64	39/64
$f(p)$	94792	1	30	1	1171	1	33	1
p	40/64	41/64	42/64	43/64	44/64	45/64	46/64	47/64
$f(p)$	4861	1	104	1	576	1	37	1
p	48/64	49/64	50/64	51/64	52/64	53/64	54/64	55/64
$f(p)$	35324	1	35	1	614	1	38	1
p	56/64	57/64	58/64	59/64	60/64	61/64	62/64	63/64
$f(p)$	6819	1	39	1	734	1	42	1

Table: Distribution of probability for $n = 7$, p for probability and $f(p)$ for number of preferences of probability p.

Fun Theorems

Theorem

There is one and only one parking preference for which the probability that every car parks is $\frac{2t-1}{2^{n-1}}$, *where* $t \in [1, 2^{n-2}]$.

Theorem

The condition of having probability $\frac{t}{2^{n-1}}, t \in \{0, 1, \ldots, 2^{n-1}\}$ *of success all have at least 1 preference satisfying.*

Thank you