

Resolving Grosswald's conjecture on GRH

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Abstract

In this paper we examine Grosswald's conjecture on $g(p)$, the least primitive root modulo p . Assuming the Generalized Riemann Hypothesis (GRH), and building on previous work by Cohen, Oliveira e Silva and Trudgian, we resolve Grosswald's conjecture by showing that $g(p) < \sqrt{p} - 2$ for all $p > 409$. Our method also shows that under GRH we have $\hat{g}(p) < \sqrt{p} - 2$ for all $p > 2791$, where $\hat{g}(p)$ is the least prime primitive root modulo p .

1 Introduction

Let $g(p)$ denote the least primitive root modulo p . Burgess [2] and Wang [9] showed that $g(p) \ll p^{1/4+\epsilon}$ for any $\epsilon > 0$. Grosswald [4] conjectured that

$$g(p) < \sqrt{p} - 2, \tag{1}$$

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for all primes $p > 409$. Clearly, the Burgess–Wang result implies (1) for all sufficiently large p . In [3] it was shown that (1) is true for all $409 < p \leq 2.5 \cdot 10^{15}$ and for all $p > 3.38 \cdot 10^{71}$; one of the ingredients in the proof is the explicit version of the Burgess character sum inequality given in [8].

In this paper, contingent on the Generalized Riemann Hypothesis (GRH) we prove (1) for the remaining values of p . In fact, we prove a stronger result. Let $\hat{g}(p)$ denote the least prime primitive root modulo p .

Theorem 1. *Assume GRH. Then $\hat{g}(p) < \sqrt{p} - 2$ for all primes $p > 2791$ and $g(p) < \sqrt{p} - 2$ for all primes $p > 409$.*

We use Theorem 1 to make the following improvement to Theorem G in [4].

Corollary 1. *Assume GRH. For all primes p , the principal congruence subgroup $\Gamma(p)$ can be generated by the matrix $\begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix}$ and $p(p-1)(p+1)/12$ canonically chosen hyperbolic elements.*

Proof. Our Theorem 1 covers the range of p in Theorem 2 in [4], whence the result follows. \square

The layout of this paper is as follows. In §2 we prove an explicit bound on the least prime primitive root. Using this, we are able to prove Theorem 1 for all $p > 10^{43}$. We introduce a sieving inequality in §3, which allows us to complete the proof of Theorem 1.

Throughout the paper we write $\omega(m)$ to denote the number of distinct prime divisors of m .

2 An explicit bound on the least prime primitive root

Theorem 2. *Assume GRH. When $p \geq 10^9$, the least prime primitive root $\hat{g}(p)$ satisfies*

$$\hat{g}(p) \leq \left(\frac{8}{5} (2^{\omega(p-1)} - 1) \log p \right)^2. \quad (2)$$

Before we prove Theorem 2 we remark that the bound in (2) is not the sharpest known. Shoup [7] has proved that

$$g(p) \ll \omega(p-1)^4 (\log(\omega(p-1)) + 1)^4 (\log p)^2,$$

and, as remarked by Martin [5, p. 279], this bound also holds for $\hat{g}(p)$. While this supersedes (2) for all sufficiently large primes p , the utility of Theorem 2 is in providing a completely explicit estimate.

We require the following result, which is easily established following the proof of Lemma 2.1 in [1].

Lemma 1 (Bach). *Let $\chi(n)$ denote a non-principal Dirichlet character modulo p . When $x \geq 1$, we have:*

$$\left| \sum_{n < x} \Lambda(n) \left(1 - \frac{n}{x}\right) - \frac{x}{2} \right| \leq \frac{\sqrt{x}}{20} + 3, \quad (3)$$

and

$$\left| \sum_{n < x} \Lambda(n) \chi(n) \left(1 - \frac{n}{x}\right) \right| \leq c(p, x) \sqrt{x} \log p, \quad (4)$$

where

$$c(p, x) := \frac{2}{3} \left(1 + \frac{2}{\sqrt{x}} + \frac{3}{x^{3/2}}\right) \left(1 + \frac{5/3}{\log p}\right) + \frac{\log x + 2}{\sqrt{x} \log p}.$$

Proof of Theorem 2. We may assume $\hat{g}(p) > 1099$, or else there is nothing to prove; indeed, one has $((8/5) \log(10^9))^2 \approx 1099.4$. As in Burgess [2, §4], we rewrite the function

$$f(n) = \begin{cases} 1 & \text{if } n \text{ is a primitive root modulo } p \\ 0 & \text{otherwise,} \end{cases}$$

as

$$f(n) = \frac{\phi(p-1)}{p-1} \left\{ 1 + \sum_{\substack{d|p-1 \\ d>1}} \frac{\mu(d)}{\phi(d)} \sum_{\substack{\chi \\ \text{ord}(\chi)=d}} \chi(n) \right\},$$

where \sum_{χ} is taken over all the $\phi(d)$ Dirichlet characters of order d modulo p . Suppose that $f(n) = 0$ for all primes n (and hence all prime powers) with $n < x$. We may assume $x \geq 1099$. We have

$$1 + \sum_{\substack{d|p-1 \\ d>1}} \frac{\mu(d)}{\phi(d)} \sum_{\substack{\chi \\ \text{ord}(\chi)=d}} \chi(n) = 0 \quad (5)$$

for all prime powers $n < x$. We multiply (5) by $\Lambda(n)(1 - n/x)$ and sum over all $n < x$. This gives

$$\sum_{n < x} \Lambda(n) \left(1 - \frac{n}{x}\right) + \sum_{\substack{d|p-1 \\ d>1}} \frac{\mu(d)}{\phi(d)} \sum_{\substack{\chi \\ \text{ord}(\chi)=d}} \sum_{n < x} \Lambda(n) \chi(n) \left(1 - \frac{n}{x}\right) = 0.$$

We apply Lemma 1 and observe that $c(p, x) \leq 7/9$ provided $p \geq 10^9$ and $x \geq 1099$. Therefore

$$\frac{x}{2} \leq \frac{\sqrt{x}}{20} + 3 + \sum_{\substack{d|p-1 \\ d>1 \\ \mu(d) \neq 0}} \frac{7}{9} \sqrt{x} \log p,$$

which implies

$$\sqrt{x} \leq \frac{1}{10} + \frac{6}{\sqrt{x}} + \frac{14}{9} (2^{\omega(p-1)} - 1) \log p \leq \frac{8}{5} (2^{\omega(p-1)} - 1) \log p.$$

The result follows. □

Corollary 2. *Theorem 1 is true except possibly when $p \in (2.5 \cdot 10^{15}, 10^{43})$ and $\omega(p-1) \in [7, 28]$.*

Proof. Unconditionally, Robin (see [6, Thm 11]) proved

$$\omega(n) \leq 1.385 \frac{\log n}{\log \log n}, \quad (n \geq 3).$$

This, when combined with Theorem 2, shows that $\hat{g}(p) < \sqrt{p} - 2$ provided $p \geq 10^{49}$. Hence we may assume $p \leq 10^{49}$. But the assumption $p \leq 10^{49}$ leads to $\omega(p-1) \leq 31$ and now Theorem 2 implies the result provided $p \leq 10^{47}$. Repeating this process, we find that $\omega(p-1) \leq 28$ and $p \leq 10^{43}$. On the other hand, in light of the computations carried out in [3, §4], we may assume $p \geq 2.5 \cdot 10^{15}$, in which case Theorem 2 proves the result provided $\omega(p-1) \leq 6$. \square

3 Computations using the sieve

In this section we follow closely the argument given in [3, §3]. Suppose e is an even divisor of $p-1$. Let p_1, \dots, p_s be the primes dividing $p-1$ that do not divide e . Set $\delta = 1 - \sum_{i=1}^s p_i^{-1}$, and set $n = \omega(p-1)$. In applying our method it is essential to choose $\delta > 0$.

Theorem 3. *Assume GRH. If $\hat{g}(p) > x$, then we have:*

$$\hat{g}(p) \leq \left(2c(p, x) \left(2 + \frac{s-1}{\delta} \right) 2^{n-s} \log p \right)^2. \quad (6)$$

We postpone the proof of Theorem 3 until §3.2. From Theorem 3 we immediately obtain the following corollary which is more readily applied.

Corollary 3. *Assume GRH. If $p \geq p_0$, then*

$$\hat{g}(p) \leq \left(C \left(2 + \frac{s-1}{\delta} \right) 2^{n-s} \log p \right)^2, \quad (7)$$

where the constant is given in Table 1.

Proof. When $p \geq p_0$, the right-hand side of (7) is bounded below by $x := ((4/3)4 \log p_0)^2$. Hence we may assume $\hat{g}(p) > x$, or else there is nothing to prove. Now Theorem 3 establishes the result with $C(p_0) := 2c(p_0, x)$. \square

Our proof of Theorem 1 will apply Theorem 3 directly, but we have included Corollary 3 as it may have application elsewhere.

Corollary 4. *Theorem 1 is true except possibly when $\omega(p-1) \in [12, 13, 14]$.*

p_0	10^2	10^4	10^6	10^8	10^{10}
C	2.1127	1.6821	1.5556	1.496	1.4614
p_0	10^{12}	10^{14}	10^{16}	10^{18}	10^{20}
C	1.4389	1.4231	1.4114	1.4023	1.3952

Table 1: Values of C for various choices of p_0

Proof. In light of Corollary 2, we may assume $7 \leq n \leq 28$ and $2.5 \cdot 10^{15} < p < 10^{43}$. We have the obvious lower bound $p - 1 \geq q_1 \dots q_n$, where q_i denotes the i th prime, and hence we may assume

$$p > \max \left\{ 1 + \prod_{i=1}^n q_i, 2.5 \cdot 10^{15} \right\}.$$

For example, when $n = 15$, this leads to $p > 6.1 \cdot 10^{17}$.

Set $x = \sqrt{p} - 2$. We may assume $\hat{g}(p) > x$ or else there is nothing to prove. Hence the conclusion of Theorem 3 holds. Now each choice of s allows us to show that

$$\hat{g}(p) < \sqrt{p} - 2 \tag{8}$$

holds when p is larger than an explicitly computable value; one simply bounds the right-hand side of (6) from above, using¹ $\delta \geq 1 - \sum_{i=n-s+1}^n p_i$. Of course, we then choose the value of s that gives the best result. For example, when $n = 15$ we find that $s = 12$ leads to

$$\delta \geq 1 - (1/7 + 1/13 + \dots + 1/47) > 0.3717$$

and therefore the right-hand side of (6) is less than $\sqrt{p} - 2$ provided $p \geq 3.2 \cdot 10^{16}$; hence any exception to (8) must satisfy $p < 3.2 \cdot 10^{16}$. Notice that because our lower and upper bounds on any potential exceptions overlap, this proves the result when $n = 15$. The best choice turns out to be $s = n - 2$ when $7 \leq n \leq 12$ and $s = n - 3$ when $13 \leq n \leq 28$. In fact, this is enough to prove (8) except when $n = 12, 13, 14$. The lower bound of $1 + \prod_{i=1}^n q_i$ does the job when $15 \leq n \leq 28$ and the lower bound of $1.6 \cdot 10^{15}$ does the job when $7 \leq n \leq 11$. \square

3.1 An algorithm

In order to deal with the cases when $n = 12, 13, 14$, we introduce an algorithm. Before giving the algorithm, we explain the main idea.

Suppose $n = 14$. Using the idea presented in the proof of Corollary 4, we find that any exception to Grosswald's conjecture must lie in the interval $(1.30 \cdot 10^{16}, 1.71 \cdot 10^{16})$. In principle one could check the conjecture directly for each prime p in this interval, but the size

¹One helpful fact — if the right-hand side of (6) is less than $\sqrt{p} - 2$ for some p , then the same is true for all larger p .

of the interval makes this prohibitive. There are $2.05 \cdot 10^{15}$ odd values of p to consider. (Of course many of these are not prime, but there are still about 10^{14} primes in this interval.) Instead, we break the problem into cases depending upon which primes divide $p - 1$. We already know that 2 divides $p - 1$, so we start with the prime 3. If 3 divides $p - 1$, then we have one third as many values of p to check, roughly $6.83 \cdot 10^{14}$ values of p . If 3 does not divide $p - 1$, then this leads to an improved lower bound on p , as well as an improved lower bound on δ and hence an improved upper bound on p ; in short, the interval under consideration shrinks. In this particular case, the interval shrinks to $(2.04 \cdot 10^{17}, 1.45 \cdot 10^{15})$, which is empty, so there is nothing to check.

Returning to the case where 3 divides $p - 1$, the number of exceptions is still quite large. However, we may consider whether 5 divides $p - 1$. We continue in this way until the number of possible values of p under consideration is less than 10^6 . At that point we go through the list and throw out all values of p except those where p is prime, $\omega(p - 1) = 14$, and $p - 1$ satisfies the given divisibility conditions (depending upon which sub-case we are considering). We append these exceptional values of p to a list and continue this recursive procedure until we have exhausted all possibilities. One can easily find the least prime primitive root for the list of exceptions via standard methods and check the conjecture directly. When $n = 14$, this algorithm takes only 5 seconds (on a 1.7 GHz MacBook Air) to complete and the list of exceptions is empty, so there is nothing further to check. The number of exceptions for other values of n is given in Table 2.

Table 2: Number of exceptions for $n = 12, 13, 14$

n	12	13	14
# of exceptions	61,114	6,916	0

For completeness, we give the pseudocode for our recursive algorithm. Suppose $X \cup Y$ consists of the first k primes for some $k \in \mathbb{Z}_{\geq 0}$. Algorithm 1 will verify $\hat{g}(p) < \sqrt{p} - 2$ when $\omega(p - 1) = n$ under the assumption that q divides $p - 1$ for all $q \in X$ and q does not divide $p - 1$ for all $q \in Y$. More precisely, rather than verifying the conjecture for all p , the algorithm will generate a manageable list of possible exceptions which can be checked individually, as described above. The sets X and Y are allowed to be empty, although in practice we may always assume $2 \in X$. (Running the algorithm with $n = 14$, $X = \{2\}$, $Y = \emptyset$ will carry out the computation described above. The Sage code is available on the website of one of the authors: http://myweb.csuchico.edu/~kmcgown/code/grosswald_GRH_code.zip.)

Proof of Theorem 1. We have implemented Algorithm 1 in Sage. Running our code on $n = 12, 13, 14$, including finding the least prime primitive root and checking the conjecture directly for the list of 68,030 exceptions, takes about 2 minutes. In light of Corollary 4, this proves the theorem. \square

Algorithm 1 Grosswald(n, X, Y)

```
1:  $L :=$  first  $n$  primes not in  $Y$ 
2:  $lower := \max\{product(L) + 1, 2 \cdot 10^{15}\}$ 
3:  $upper := 0$ 
4: for  $s \in \{1, \dots, n - 1\}$  do
5:    $M :=$  largest  $s$  primes in  $L$ 
6:    $\delta := 1 - \sum_{q \in M} \frac{1}{q}$ 
7:   if  $\delta \leq 0$  then
8:     continue
9:   end if
10: Choose  $p$  large enough so that when  $x = \sqrt{p} - 2$ , we have:
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$$\left(2c(p, x) \left(2 + \frac{s-1}{\delta}\right) 2^{n-s} \log(p)\right)^2 < x$$

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11: if  $upper = 0$  or  $p < upper$  then
12:    $upper := p; \hat{s} := s; \hat{\delta} := \delta; \hat{M} := M$ 
13: end if
14: end for
15:  $print(n, \hat{s}, \hat{\delta}, \hat{M}, lower, upper)$ 
16:  $prodX := product(X)$ 
17:  $enum := (upper - lower)/prodX$ 
18: if  $enum \leq 0$  then
19:    $print(\text{"Nothing more to check."})$ 
20: else if  $enum > 10^6$  then
21:    $print(\text{"Break into cases."})$ 
22:    $q :=$  smallest prime not in  $X \cup Y$ 
23:    $Grosswald(n, X \cup \{q\}, Y)$ 
24:    $Grosswald(n, X, Y \cup \{q\})$ 
25: else
26:    $print(\text{"Find all exceptions."})$ 
27:    $I := [(lower - 1)/prodX, (upper - 1)/prodX] \cap \mathbb{Z}$ 
28: Remove the elements of  $I$  divisible by primes in  $Y$ 
29: for  $k \in I$  do
30:    $p := k * prodX + 1$ 
31:   if  $is\_prime(p)$  and  $length(prime\_divisors(p - 1)) = n$  then
32:     append  $p$  to global list of exceptions
33:   end if
34: end for
35: end if
```

3.2 Proof of Theorem 3

Let p be an odd prime. Let e be an even divisor of $p - 1$. We say that n is e -free if the equation $y^d \equiv n \pmod{p}$ is insoluble for all divisors d of e with $d > 1$. An integer is a primitive root if and only if it is $(p - 1)$ -free. Note also that being e -free depends only on the radical of e , that is, on the product of the distinct prime divisors of e . We define the function

$$f_e(n) = \begin{cases} 1 & \text{if } n \text{ is } e\text{-free} \\ 0 & \text{otherwise.} \end{cases}$$

Define the multiplicative function $\theta(n) = \phi(n)/n$, where $\phi(n)$ is Euler's totient function. For the rest of the proof we will need the following:

Lemma 2.

$$f_e(n) = \theta(e) \left\{ 1 + \sum_{\substack{d|e \\ d>1}} \frac{\mu(d)}{\phi(d)} \sum_{\text{ord}(\chi)=d} \chi(n) \right\}.$$

Proof. Let g be a primitive root modulo p . Let $n = g^v$, and let $e(t) = e^{2\pi it}$. Then

$$\sum_{\substack{\chi \\ \text{ord}(\chi)=d}} \chi(n) = \sum_{\substack{1 \leq k \leq d \\ \gcd(k,d)=1}} e\left(\frac{kv}{d}\right) = \sum_{t|d} \mu(t) \sum_{k=1}^{d/t} e\left(\frac{vtk}{d}\right).$$

The inner sum is 0 whenever $\frac{tv}{d} \notin \mathbb{Z}$ and is $\frac{d}{t}$ otherwise. Let $m = \gcd(d, v)$. Since $\mu(ab) = \mu(a)\mu(b)$ when $\gcd(a, b) = 1$ and 0 otherwise, then the sum is

$$\begin{aligned} \sum_{\substack{\chi \\ \text{ord}(\chi)=d}} \chi(n) &= \sum_{t|m} \mu\left(t \cdot \frac{d}{m}\right) \frac{m}{t} = \sum_{\substack{t|m \\ \gcd(t, \frac{d}{m})=1}} \mu(t) \mu\left(\frac{d}{m}\right) \frac{m}{t} \\ &= \mu\left(\frac{d}{m}\right) m \sum_{\substack{t|m \\ \gcd(t, \frac{d}{m})=1}} \frac{\mu(t)}{t} = \mu\left(\frac{d}{m}\right) m \prod_{\substack{p|m \\ p \nmid \frac{d}{m}}} \left(1 - \frac{1}{p}\right) = \frac{\phi(d)}{\phi\left(\frac{d}{m}\right)} \mu\left(\frac{d}{m}\right). \end{aligned}$$

When n is e -free, we have $\gcd(v, d) = 1$ for all $d|e$. Therefore

$$1 + \sum_{\substack{d|e \\ d>1}} \frac{\mu(d)}{\phi(d)} \sum_{\text{ord}(\chi)=d} \chi(n) = \sum_{d|e} \frac{\mu(d)}{\phi(d)} \left(\frac{\phi(d)}{\phi(d)} \mu(d) \right) = \sum_{d|e} \frac{\mu(d)^2}{\phi(d)} = \prod_{p|e} \left(1 + \frac{1}{p-1}\right) = \frac{e}{\phi(e)}.$$

When n is not e -free, then $M = \gcd(e, v) > 1$. We may assume e is squarefree. Then

$$1 + \sum_{\substack{d|e \\ d>1}} \frac{\mu(d)}{\phi(d)} \sum_{\text{ord}(\chi)=d} \chi(n) = \sum_{d|e} \frac{\mu(d) \mu\left(\frac{d}{(d,v)}\right)}{\phi\left(\frac{d}{(d,v)}\right)} = \sum_{d|\frac{e}{M}} \sum_{k|M} \frac{\mu(dk) \mu(d)}{\phi(d)} = \sum_{d|\frac{e}{M}} \frac{\mu^2(d)}{\phi(d)} \sum_{k|M} \mu(k) = 0.$$

The last equality follows from $\sum_{d|n} \mu(d) = 0$ for all $n > 1$. □

Then, recalling the definitions of s, p_i and δ from the start of §3, we see

$$\sum_{i=1}^s f_{p_i e}(n) - (s-1)f_e(n) \begin{cases} = 1 & \text{if } n \text{ is } (p_i e)\text{-free for all } i \\ \leq 0 & \text{otherwise,} \end{cases}$$

whence

$$\begin{aligned} f_{p-1}(n) &\geq \sum_{i=1}^s f_{p_i e}(n) - (s-1)f_e(n) \\ &= \sum_{i=1}^s (f_{p_i e}(n) - \theta(p_i)f_e(n)) + \delta f_e(n). \end{aligned} \quad (9)$$

Observe that

$$\begin{aligned} f_{p_i e}(n) - \theta(p_i)f_e(n) &= \theta(p_i e) \sum_{\substack{d|p_i e \\ d|e \\ d>1}} \frac{\mu(d)}{\phi(d)} \sum_{\substack{\chi \\ \text{ord}(\chi)=d}} \chi(n) \\ &= \theta(p_i e) \sum_{d|e} \frac{\mu(p_i d)}{\phi(p_i d)} \sum_{\substack{\chi \\ \text{ord}(\chi)=p_i d}} \chi(n). \end{aligned} \quad (10)$$

Inserting (10) into (9) leads to

$$1 + \frac{1}{\delta} \sum_{i=1}^s \theta(p_i) \sum_{d|e} \frac{\mu(p_i d)}{\phi(p_i d)} \sum_{\substack{\chi \\ \text{ord}(\chi)=p_i d}} \chi(n) + \sum_{\substack{d|e \\ d>1}} \frac{\mu(d)}{\phi(d)} \sum_{\substack{\chi \\ \text{ord}(\chi)=d}} \chi(n) \leq \frac{f_{p-1}(n)}{\delta \theta(e)}. \quad (11)$$

Suppose $f_{p-1}(n) = 0$ for all primes n (and hence all prime powers) with $n < x$. We multiply (11) by $\Lambda(n)(1 - n/x)$, sum over all $n < x$, which yields

$$\begin{aligned} &\sum_{n<x} \Lambda(n) \left(1 - \frac{n}{x}\right) \\ &+ \frac{1}{\delta} \sum_{i=1}^s \theta(p_i) \sum_{d|e} \frac{\mu(p_i d)}{\phi(p_i d)} \sum_{\substack{\chi \\ \text{ord}(\chi)=p_i d}} \sum_{n<x} \chi(n) \Lambda(n) \left(1 - \frac{n}{x}\right) \\ &+ \sum_{\substack{d|e \\ d>1}} \frac{\mu(d)}{\phi(d)} \sum_{\substack{\chi \\ \text{ord}(\chi)=d}} \sum_{n<x} \chi(n) \Lambda(n) \left(1 - \frac{n}{x}\right) \leq 0. \end{aligned}$$

We write

$$\sum_{n<x} \Lambda(n) \left(1 - \frac{n}{x}\right) = \frac{x}{2} + G(x),$$

and use the estimates in (3) and (4) to obtain

$$\begin{aligned}
& \frac{x}{2} + G(x) \\
& \leq \frac{1}{\delta} \sum_{i=1}^s \theta(p_i) \sum_{\substack{d|e \\ \mu(p_i d) \neq 0}} c(p, x) \sqrt{x} \log p + \sum_{\substack{d|e \\ d > 1 \\ \mu(d) \neq 0}} c(p, x) \sqrt{x} \log p \\
& \leq c(p, x) \sqrt{x} \log p \left[\left(1 + \frac{1}{\delta} \sum_{i=1}^s \theta(p_i) \right) 2^{n-s} - 1 \right] \\
& = c(p, x) \sqrt{x} \log p \left[\left(2 + \frac{s-1}{\delta} \right) 2^{n-s} - 1 \right].
\end{aligned}$$

This leads to

$$\begin{aligned}
\sqrt{x} & \leq \frac{2|G(x)|}{\sqrt{x}} + 2c(p, x) \left[\left(2 + \frac{s-1}{\delta} \right) 2^{n-s} - 1 \right] \log p \\
& \leq \frac{1}{10} + \frac{6}{\sqrt{x}} + 2c(p, x) \left[\left(2 + \frac{s-1}{\delta} \right) 2^{n-s} - 1 \right] \log p \\
& \leq 2c(p, x) \left(2 + \frac{s-1}{\delta} \right) 2^{n-s} \log p - \frac{191}{90} - \frac{4}{3} \log p.
\end{aligned}$$

The result follows. □

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