On generalizing happy numbers to fractional base number systems

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Abstract

Let n be a positive integer and $S_2(n)$ be the sum of the squares of its decimal digits. When there exists a positive integer k such that the k-th iterate of S_2 on n is 1, i.e., $S_2^k(n) = 1$, then n is called a happy number. The notion of happy numbers has been generalized to different bases, different powers and even negative bases. In this article we consider generalizations to fractional number bases. Let $S_{e,\frac{p}{q}}(n)$ be the sum of the e-th powers of the digits of n base p/q. Let k be the smallest nonnegative integer for which there exists a positive integer $m > k$ satisfying $S_{e,\frac{p}{q}}^k(n) = S_{e,\frac{p}{q}}^m(n)$. We prove that such a k, called the height of n, exists for all n, and that, if $q = 2$ or $e = 1$, then k can be arbitrarily large.

1 Introduction

Let n be a positive integer and $S_2(n)$ be the sum of the squares of its decimal digits. It is well known (for a complete proof look at [6]) that if you apply a sufficiently high iterate of S_2 to n, the result is either 1 or is in the cycle

$$4 \to 16 \to 37 \to 58 \to 89 \to 145 \to 42 \to 20 \to 4.$$

If the iteration reaches 1, we say n is *happy*. A natural generalization is to allow for any base $b \geq 2$ representation of the digits and to replace sum of squares of digits, with the sum of e-th powers of the digits for some integer $e \geq 1$. Let $S_{e,b}(n)$ be the sum of e-th powers of the digits of n when n is written in base b. If there exists an integer k such that $S_{e,b}^k(n) = 1$, we say n is an e-power b-happy number (when $e = 2$, we call n a b-happy number). Suppose that there exist integers k and m with $0 \leq k < m$ such that $S_{e,b}^k(n) = S_{e,b}^m(n)$, then the iterates of n under $S_{e,b}$ will cycle through the sequence $\{S_{e,b}^k(n), S_{e,b}^{k+1}(n), \ldots, S_{e,b}^{m-1}(n)\}$. If $m - k$ is minimal, then we say that n reaches the cycle $(S_{e,b}^k(n), S_{e,b}^{k+1}(n), \ldots, S_{e,b}^{m-1}(n))$. If k is the smallest non-negative integer for which this is true, we say k is the *height* of n.

The study of which cycles can be reached for $e \in \{2, 3\}$ and $2 \leq b \leq 10$ has been done by Grundman and Teeple in [3]. The techniques in [3] can easily be used to find the cycles for other choices of e and b. Another generalization is to allow the base b to be a negative number. It turns out that for a positive integer n, there is a unique set of digits $0 \leq a_i \leq |b| - 1$ such that $n = \sum_{i=0}^{r} a_i b^i$. Grundman and Harris, in [5], find the cycles reached for $-2 \geq b \geq -10$ and $e = 2$. The authors also study in what cases there exist consecutive b-happy numbers in an arithmetic progression, generalizing work of El-Sedy and Siksek [2] and the work of Grundman and Teeple [4].

In [1], Bland et al. addressed a series of questions regarding a generalization of happy numbers to fractional bases. For integers $p > q > 0$ with $\gcd(p, q) = 1$, each positive integer

n has a unique representation in base p/q. Namely, there exists an integer $r \geq 0$ such that for every integer $i \in \{0, 1, \ldots, r\}$ there exists an integer $a_i \in \{0, 1, \ldots, p-1\}$ with $a_r \neq 0$ and

$$n = \sum_{i=0}^{r} a_i \left(\frac{p}{q}\right)^i.$$

For our notation, we will say $n = \overline{a_r a_{r-1} \cdots a_1 a_0}_{\frac{p}{q}}$. Let $S_{e,\frac{p}{q}}(n)$ be the sum of the e-th powers of the digits of n when written in fractional base p/q, i.e.,

$$S_{e,\frac{p}{q}}(n) = \sum_{i=0}^{r} a_i^e.$$

In [1], the authors studied the case when $e = 2$ and proved that there are no happy numbers greater than 1 for any fractional base. They mainly study the fractional base $3/2$, finding the possible cycles that $S_{2,\frac{3}{2}}$ can reach. They end the paper with several questions. The three we will focus on in this paper are the following:

1. Can we find the cycles reached by $S_{e,b}$ for different e-th powers when $p/q = 3/2$?
2. Can we find the cycles reached by $S_{e,b}$ for different p/q when we restrict to $e = 2$?
3. Are there positive integers n of arbitrarily large height?

In the case of positive integer bases, that there are numbers with arbitrary height is relatively simple to prove because you can find an n such that $S_{e,b}(n) = k$ for any positive integer k by having n be a number with k 1's in its base b expansion. For example, let $a_1 = 10$, then a_1 has height 1 since $S_2(10) = 1$. Let

$$a_2 = \underbrace{11 \cdots 1}_{10}.$$

Since $S_2(a_2) = 10$, a_2 has height 2. Let

$$a_n = \underbrace{11 \cdots 1}_{a_{n-1}}.$$

Then a_n has height n. This simple process creates a sequence of numbers with larger and larger heights by attaching the appropriate number of 1's to a number. The problem with fractional bases is that not every choice of digits leads to an integer. For example $\overline{11}_{\frac{3}{2}}$ is not an integer, since $1 + \frac{3}{2} \notin \mathbb{Z}$.

We answer the three questions with two theorems. The first theorem answers two of the questions.

Theorem 1. *Let $p > q$ be positive integers with $\gcd(p, q) = 1$, and let e be a positive integer. Then, for every positive integer n, the repeated iterations of the function $S_{e,\frac{p}{q}}$ on n will eventually reach a cycle. In particular, the possible cycles reached for $1 \leq e \leq 12$, $p/q = 3/2$ can be found in Table 2, answering the first question. Also, the possible cycles reached for $e \in \{2, 3, 4\}$ and $p/q \in \{5/2, 5/3, 5/4, 7/2\}$ are in Table 3, answering the second question.*

The second theorem answers the third question for a special class of fractional bases that includes $3/2$, and for all fractional bases when $e = 1$.

Theorem 2. *Let $p > q$ be positive integers with $\gcd(p, q) = 1$, and let e and H be positive integers. If $q = 2$ or $e = 1$, then there exists an integer n such that the height of n is H.*

In Section 2, we will present useful background on fractional base number systems. In Section 3, we prove Theorem 1. Finally, in Section 4, we prove Theorem 2.

2 Fractional base number systems

As mentioned in the introduction, for any p/q with $\gcd(p, q) = 1$ and $p > q$, for every positive integer n, there exist *fractional digits* $a_0, a_1, \ldots, a_r$ satisfying $0 \leq a_i < p$ for $i \in \{0, 1, \ldots, r-1\}$ and $0 < a_r < p$ such that

$$n = \sum_{i=0}^{r} a_i \left(\frac{p}{q}\right)^i.$$

We will use the following notation to denote that a_i are the fractional digits of n base p/q.

$$n = \overline{a_r a_{r-1} a_{r-2} \ldots a_2 a_1 a_0}_{\frac{p}{q}}.$$

For example base 3/2 uses numbers 0, 1, 2 as digits. Table 1 is a table of numbers in base 3/2:

n	n in base 3/2	n	n in base 3/2
0	$\overline{0}_{\frac{3}{2}}$	6	$\overline{210}_{\frac{3}{2}}$
1	$\overline{1}_{\frac{3}{2}}$	7	$\overline{211}_{\frac{3}{2}}$
2	$\overline{2}_{\frac{3}{2}}$	8	$\overline{212}_{\frac{3}{2}}$
3	$\overline{20}_{\frac{3}{2}}$	9	$\overline{2100}_{\frac{3}{2}}$
4	$\overline{21}_{\frac{3}{2}}$	10	$\overline{2101}_{\frac{3}{2}}$
5	$\overline{22}_{\frac{3}{2}}$	11	$\overline{2102}_{\frac{3}{2}}$

Table 1: The first 12 non-negative integers in the 3/2 base number system.

It is easy to find n given its expansion in base p/q, but going the other way around is a little harder. Suppose we have the number n and we want to find its fractional digits base p/q. Let $n = \overline{a_r a_{r-1} \cdots a_1 a_0}_{\frac{p}{q}}$. Then

$$n - a_0 = \left(\frac{p}{q}\right) \overline{a_r a_{r-1} \cdots a_1}_{\frac{p}{q}}.$$

The left side is an integer, so the right side is also an integer. Since $\gcd(p, q) = 1$, $q | \overline{a_r a_{r-1} \cdots a_1}_{\frac{p}{q}}$, and so $p | (n - a_0)$. Therefore $n \equiv a_0 \bmod p$. There is a unique a_0 in $\{0, 1, 2, \cdots, p-1\}$ that is congruent to n modulo p. But we also have

$$\overline{a_r a_{r-1} \cdots a_1}_{\frac{p}{q}} = \left(\frac{q}{p}\right)(n - a_0).$$

We repeat the process and we can say that

$$n \equiv \left(\frac{q}{p}\right)(n - a_0) - a_1 \bmod p.$$

Therefore, we can find a_1. We can repeat this process until we reach 0 and find all of the digits of n.

We can summarize the algorithm to translate numbers into the fractional base $\frac{p}{q}$ as follows:

1. $n_0 = n \pmod p$.
2. $n = (n - n_0)\left(\frac{q}{p}\right)$.
3. Repeat steps 1 and 2, until n is zero.

As an example, suppose we want to find the digits of 12 in base 3/2. First we have $12 \equiv 0 \bmod 3$, so $a_0 = 0$. Then we calculate $(12-0)\frac{2}{3} = 8$. We find $8 \equiv 2 \bmod 3$, so $a_1 = 2$. Then we find $(8-2)\frac{2}{3} = 4$ and $4 \equiv 1 \bmod 3$, so $a_2 = 1$. Then we find $(4-1)(2/3) = 2$ and $2 \equiv 2 \bmod 3$, so $a_3 = 2$. Since the next step yields 0, we've found that $12 = \overline{2120}_{\frac{3}{2}}$.

3 The cycles formed when iterating $S_{e,\frac{3}{2}}$

An integer $n > 1$ cannot be *happy* in a fractional base number system. Indeed suppose that n is e-power p/q-happy, then $S^m_{e,\frac{p}{q}}(n) = 1$ for some minimal positive integer m. But then $k = S^{m-1}_{e,\frac{p}{q}}(n)$ must satisfy that the sum of the e-th powers of its digits is 1. Therefore the fractional base expansion of k is $\overline{100\cdots 0}_{\frac{p}{q}}$. But that means $k = (p/q)^r$ for some integer r. This number is not an integer unless $r = 0$, which would imply $k = 1$, but we assumed $k > 1$. While happiness is impossible, we can still search which cycles can be reached. For us to be able to prove that the determination of cycles is complete, we need to first prove the following lemma.

Lemma 1. *Let p/q satisfy $p > q$ and $\gcd(p,q) = 1$, and let e be a positive integer. Then, there exists an n^* such that $S_{e,\frac{p}{q}}(n^*) \geq n^*$, and $S_{e,\frac{p}{q}}(n) < n$ for all $n > n^*$.*

The values of n^* for different values of e and $p/q = 3/2$ can be found in the last column of Table 2. The values of n^* for $e \in \{2,3,4\}$ and $p/q \in \{5/2, 5/3, 5/4, 7/2\}$ are in Table 3.

Proof. Let n be a positive integer. Then

$$n = \overline{a_r a_{r-1}\cdots a_1 a_0}_{\frac{p}{q}} = \sum_{i=0}^{r} a_i \left(\frac{p}{q}\right)^i \geq a_r \left(\frac{p}{q}\right)^r \geq \left(\frac{p}{q}\right)^r,$$

so $r \leq \log_{\frac{p}{q}}(n)$. But then

$$S_{e,\frac{p}{q}}(n) = \sum_{i=0}^{r} a_i^e < \sum_{i=0}^{r} p^e = (r+1)p^e \leq (\log_{\frac{p}{q}}(n) + 1)p^e.$$

Since p^e is a constant, then for a large enough n,

$$n > (\log_{\frac{p}{q}}(n) + 1)p^e > S_{e,\frac{p}{q}}(n). \tag{1}$$

Indeed, one can confirm with L'Hopital's rule that $n/(C\log(n)) \to \infty$ as $n \to \infty$ for any constant $C > 0$.

e	Cycles	n^*
1	(1), (2)	2
2	(1), (5, 8, 9)	8
3	(1), (9), (10), (17, 18)	32
4	(1), (51), (52)	77
5	(1), (131), (98, 99)	185
6	(1), (197, 260, 387, 323, 263, 450), (324, 131, 259)	419
7	(1), (771, 516, 643, 518)	1211
8	(1), (1539, 775, 1284), (1287, 1794, 1796, 2052), (1032), (1033)	2723
9	(1), (2566), (2565)	6557
10	(1), (10247)	13118
11	(1), (14342, 16388, 14344), (14341), (14340)	27968
12	(1), (28678), (28677)	62933

Table 2: Cycles reached when iterating $S_{e,\frac{3}{2}}$, and the value of n^* for different values of e.

Therefore, there is a maximum n^* such that $n^* < S_{e,\frac{p}{q}}(n^*)$.

To calculate the precise value of n^*, we use a computer to find an N for which (1) is satisfied. Then we evaluate $S_{e,\frac{p}{q}}(n)$ for all $n \leq N$ and record which one is the largest satisfying that $n \leq S_{e,\frac{p}{q}}(n)$.

□

Proof of Theorem 1. To simplify notation, let $S(n) = S_{e,\frac{p}{q}}(n)$ for all positive integers n. Let n^* be as in Lemma 1. Now, for each $m \leq n^*$, compute $m, S(m), S(S(m)), \ldots$ until it cycles. The process terminates because $S(n) < n$ for all $n > n^*$. Therefore, for $n > n^*$, there exists a positive integer k such that $S^k(n) \leq n^*$. This implies that the cycle n reaches is one that was already computed. Therefore, we need only find the cycles reached for $m \leq n^*$. The outcome of performing these calculations for different values of e and $p/q = 3/2$ is recorded in Table 2. The outcome of performing these calculations on $e \in \{2, 3, 4\}$ with $p/q \in \{5/2, 5/3, 5/4, 7/2\}$ is recorded in Table 3.

□

4 Arbitrary Heights in fractional base number systems

The key to our proof of Theorem 2 is showing that for each sufficiently large k, there exists a positive integer n such that $S_{e,\frac{p}{q}}(n) = k$. The following lemma handles the case when $q = 2$.

Lemma 2. *Let $e \geq 1$ and $p > 2$ be an odd positive integer. For every integer $k \geq 2^e$, there exists an even integer n, such that $S_{e,\frac{p}{2}}(n) = k$.*

Proof. We will prove the lemma by induction on k. To show that it is true for $k = 2^e$, consider the number 2. 2 is $\overline{2}_{\frac{p}{2}}$, therefore $S_{e,\frac{p}{2}}(2) = 2^e$. Now let $k \geq 2^e$ and assume that there exists an even m such that $S_{e,\frac{p}{2}}(m) = k$. Let $m = 2^b c$ where $b \geq 1$ and c is odd. Write m in base $p/2$ as

$$m = \overline{a_r a_{r-1} \cdots a_1 a_0}.$$

p/q	$e=2$	$e=3$	$e=4$
5 / 2	(16, 6, 5, 4), (32, 24, 29); $n^*=39$	(65), (163, 190, 73, 118, 64), (81), (80), (66), (17); $n^*=239$	(371, 276, 275, 274), (355, 130, 113), (195, 353); $n^*=1039$
5 / 3	(34, 50), (25), (26), (59), (23), (11), (10); $n^*=59$	(100, 38, 64, 102, 46), (101, 39), (127, 107, 73, 135), (162), (193), (190, 166, 218), (199, 237); $n^*=284$	(772, 804, 454, 788, 950, 658, 934, 1126, 1028, 1202, 868, 936, 390), (1027, 1137, 1125), (1122, 994), (1299), (101), (100); $n^*=1324$
5 / 4	(66, 55), (50), (58, 75, 49, 56, 67), (74, 83), (51); $n^*=74$	(311, 251, 247, 231, 371), (361), (417), (374), (360), (314), (424, 418, 436, 272, 328, 364); $n^*=464$	(1786, 1880, 1403, 1594, 1659, 2011, 2075, 1579, 2057, 1947, 1688, 1229, 1641, 1676, 1946, 1673, 1851, 1592, 1419, 1974, 2058, 2012, 2090); $n^*=2639$
7 / 2	(25, 52), (97); $n^*=97$	(341, 591, 376, 143, 187, 216, 352, 25, 280, 244, 469, 63, 128, 44, 141, 161, 197, 73, 307, 467, 377, 234, 182, 91), (35), (288, 343, 9, 16, 72), (36), (189), (190), (468); $n^*=615$	(914, 2065, 1953, 1538, 2819, 2690, 2210, 1507, 1491, 2610, 1856, 1348, 1666, 259, 1808, 2659, 3136, 1824), (1634, 1731, 994), (371, 34, 1313), (130, 354, 289, 1938, 3265, 2930, 1474, 1570), (451, 195, 2177, 1554, 179, 513, 2034, 2530); $n^*=5417$

Table 3: Cycles reached when iterating $S_{e,\frac{p}{q}}$, and the value of n^* for different values of e and p/q.

Then

$$\left(\frac{p}{2}\right)^b m + 1 = \overline{a_r a_{r-1} \cdots a_1 a_0 \underbrace{0 \cdots 0}_{b-1} 1},$$

where there are $b-1$ zero digits. Since $m = 2^b c$, $(p/2)^b m + 1$ is even. Furthermore, since it has the same digits as before with $b-1$ zeroes added and one 1 added, the sum of the e-th powers of the digits is $k+1$.

□

The following lemma handles the $e = 1$ case.

Lemma 3. *Let $p/q > 1$ be written in lowest terms. For every integer $k \geq q$, there exists n, such that $S_{1,\frac{p}{q}}(n) = k$.*

Proof. We prove by induction on t that for each $k \in \{q, q+1, \ldots, qt\}$, there exists an m_k such that $S_{1,\frac{p}{q}}(m_k) = k$. The fact that $S_{1,\frac{p}{q}}(q) = q$ proves the case of $t = 1$. Now, fix $t \geq 1$ and assume that for each $k \in \{q, q+1, \ldots, qt\}$, there exists an m_k such that $S_{1,\frac{p}{q}}(m_k) = k$. Write m_{qt} as $m_{qt} = q^\alpha b$ for some $\alpha \geq 1$ and b relatively prime to q. Suppose $m_{qt} = \overline{a_r a_{r-1} \cdots a_0}_{\frac{p}{q}}$.

Then

$$\ell = \left(\frac{p}{q}\right)^{\alpha} m_{qt} = \overline{a_r \cdots a_0 \underbrace{0 \cdots 0}_{\alpha}}.$$

We know $\ell \not\equiv 0 \bmod q$. Let w be the smallest positive integer such that $\ell + w \equiv 0 \bmod q$. Then $1 \leq w \leq q - 1 < p - 1$. But then

$$\ell + w = \overline{a_r \cdots a_0 \underbrace{0 \cdots 0}_{\alpha - 1} w},$$

because $w < p - 1$. This implies that the digital sum base p/q of the numbers $\ell + 1, \ell + 2, \ldots, \ell + w$ are $qt + 1, qt + 2, \ldots, qt + w$, respectively. Now $\ell + w$ is a multiple of q with $S_{1,\frac{p}{q}}(\ell + w) = qt + w \geq qt + 1$, and we have that for all $q \leq k \leq qt + w$, there exists m_k such that $S_{1,\frac{p}{q}}(m_k) = k$. Since $q|(\ell + w)$ and $\ell + w \geq qt + 1$, then $\ell + w \geq q(t + 1)$. Therefore, we've proved that for every $q \leq k \leq q(t + 1)$, there is an m_k such that $S_{1,\frac{p}{q}}(m_k) = k$.

□

Using these two Lemmas, we can now present the proof of Theorem 2.

Proof of Theorem 2. We will prove it by induction. Let n^* be as defined in Lemma 1. Since the cycles that are reached by iterations of $S_{e,\frac{p}{q}}$ are finite and there are finitely many of them, there is a largest integer K with height 0. Let n be an integer greater than $M = \max\{n^*, 2^e, q, K\}$. Since $n > K$, n has some height $h > 0$. Then, $S_{e,\frac{p}{q}}(n)$ has height $h - 1$, $S^2_{e,\frac{p}{q}}(n)$ has height $h - 2$, …, $S^{h-1}_{e,\frac{p}{q}}(n)$ has height 1. Therefore, for every positive integer $i \leq h$, there exists an integer n of height i.

Let $H \geq h$. Suppose that there is an integer $n > K$ with height H. Since $n > 2^e$, by Lemma 2, if $q = 2$, then there exists t such that $S_{e,\frac{p}{2}}(t) = n$. Since $n > q$, by Lemma 3, if $e = 1$, then there exists t such that $S_{1,\frac{p}{q}}(t) = n$. Therefore, in either case ($q = 2$ or $e = 1$), there exists an integer t such that $S_{e,\frac{p}{q}}(t) = n$. But $t > n^*$, which implies that $n = S_{e,\frac{p}{q}}(t) < t$. Therefore $t > n > K$ and t has height $H + 1$.

□

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