A St. Petersburg like coin flip paradox

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The following question has the flavor of the famous St. Petersburg paradox[1]:
A fair coin is flipped and the player is paid \$2 if after two flips the coin landed heads once and tails once, \$4 if it takes four coin flips to get the same number of heads as tails, and in general, \$$2k$ if it takes $2k$ flips for the coin to land the same number of times heads as tails. How much would a person be willing to pay to play this game?
Let X be the random variable representing this number. Like in the St. Petersburg paradox, you get that the expected payoff diverges to infinity. In this short article, we'll prove that $\mathbb{E}[X] = \infty$ by first calculating the probability that $X = 2k$ for a positive integer k (note that X is always even).

Before we get to the theorem, let's remember a classical combinatorics result (which can be found in [9]):

Lemma 1. *The number of paths from $(0,0)$ to (n,n) using only unit steps to the right or up that don't go above the diagonal from $(0,0)$ to (n,n) is the n-th Catalan number:*

$$\frac{1}{n+1}\binom{2n}{n}.$$

Theorem 1. *Let k be a positive integer. Then*

$$\mathbb{P}[X = 2k] = \frac{2}{4^k k}\binom{2k-2}{k-1}.$$

Proof. We want to figure out in how many ways we can get k heads and k tails without having the number of heads equal the number of tails happen beforehand. We can think of this as the number of paths from $(0,0)$ to (k,k) using only steps to the right ("heads") and steps upwards ("tails") in such a way that it doesn't intersect the diagonal from $(0,0)$ to (k,k). This is almost the situation of Lemma 1, however in this case we can't reach a situation where we touch the diagonal, while the case in the Lemma allows "touching" the diagonal as long as it's not crossed.

If the first coin flip is heads, then we're at $(1,0)$. To avoid touching the diagonal from $(0,0)$ to (k,k), we can consider the number of paths from $(1,0)$ to $(k,k-1)$ that don't cross the diagonal from $(1,0)$ to $(k,k-1)$. But considering the subsquare of side-length $k-1$ with vertices $(1,0),(1,k-1),(k,0),(k,k-1)$ reveals that, by Lemma 1, the number of such paths is the $(k-1)-th$ Catalan number, i.e., $\frac{1}{k}\binom{2k-2}{k-1}$. Given that we could have gone "up" first, we multiply this number of paths by 2. Given that there are $2^{2k} = 4^k$ possible outcomes of flipping a coin $2k$ times, the formula for $\mathbb{P}[X = 2k]$ follows. □

Remark 1. *This result is known as the "first return on a symmetric one dimensional random walk". Most proofs I found of this result (see [4], [5]) use generating functions. I think this proof is easier.*

Now note that

$$\frac{1}{4^{n-1}}\binom{2n-2}{n-1} - \frac{1}{4^n}\binom{2n}{n} = \frac{2}{4^n n}\binom{2n-2}{n-1},$$

[1]In [3], the paradox is written as follows: "A fair coin is flipped and the player is paid \$2 if a head occurs on the first toss, \$4 if a head first appears on the second toss and, in general, \$$2^k$ if heads first appears on the k-th toss. What would you be willing to pay for the privilege of playing this game?" The paradox was introduced by Daniel Bernoulli in [2] and has been heavily studied in economics over the years, some references include [6], [8], and [1]

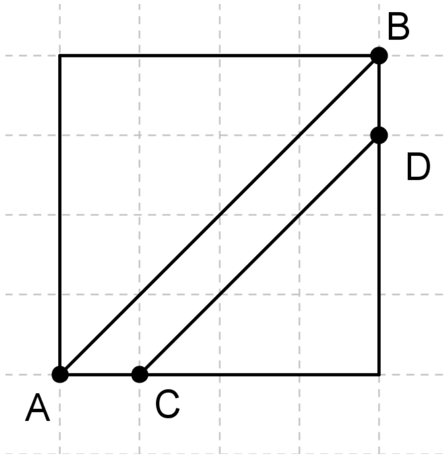

Figure 1: Example of $k = 4$. The diagonal $\overline{AB}$ is to be avoided, the diagonal $\overline{CD}$ is not to be crossed.

and so

$$\sum_{k=1}^{\infty} \frac{2}{4^k k}\binom{2k-2}{k-1} = \sum_{k=1}^{\infty}\left(\frac{1}{4^{k-1}}\binom{2k-2}{k-1} - \frac{1}{4^k}\binom{2k}{k}\right) = \sum_{k=0}^{\infty}\frac{1}{4^k}\binom{2k}{k} - \sum_{k=1}^{\infty}\frac{1}{4^k}\binom{2k}{k} = 1.$$

The reason it equals 1 is that it telescopes, and this is legal because

$$\lim_{k\to\infty}\frac{1}{4^k}\binom{2k}{k} = 0.$$

But this implies that

Corollary 1. *The probability that eventually the number of heads will equal the number of tails is 1. In other words, the probability that the process terminates is 1, which implies* $\mathbb{P}[X = \infty] = 0$.

Proof. From probability, we know that the sum of the probability of all individual events is 1. But then

$$1 = \sum_{k=1}^{\infty}\mathbb{P}[X = 2k] + \mathbb{P}[X = \infty] = 1 + \mathbb{P}[X = \infty].$$

Therefore, $\mathbb{P}[X = \infty] = 0$. □

Remark 2. *This corollary was first proven by Pólya in [7] in a more general setting.*

We are now ready to prove our main result:

Corollary 2 (Coin Flip Paradox)**.**

$$\mathbb{E}[X] = \infty.$$

Proof. Since $\mathbb{P}[X = \infty] = 0$, then the expected value of X is

$$\mathbb{E}[X] = \sum_{k=1}^{\infty}(2k)\mathbb{P}[X = 2k] = \sum_{k=1}^{\infty}(2k)\frac{2}{4^k k}\binom{2k-2}{k-1} = \sum_{k=1}^{\infty}\frac{1}{4^{k-1}}\binom{2k-2}{k-1} = \sum_{n=0}^{\infty}\frac{1}{4^n}\binom{2n}{n}.$$

Since the sum of the $2n$-th row of Pascal's triangle is 4^n, then $\binom{2n}{n} \geq \frac{1}{2n+1}4^n$. Therefore

$$\mathbb{E}[X] \geq \sum_{n=0}^{\infty}\frac{1}{2n+1} = \infty.$$

□

Remark 3. *Using Stirling's formula, since* $\binom{2n}{n} \sim \frac{4^n}{\sqrt{\pi n}}$*, we could show that even if the payout was* $\sqrt{k}$ *instead of* $2k$*, the expected value would diverge to infinity. However, if the payout was* $\sqrt{k}/\log^2 k$*, then the payout would converge.*

Many players playing the game

Suppose you have a class of n students and you ask them to flip a coin until they have the same number of heads as tails. As shown above, the expected value of coin flips is infinite. However, in practice, the process will end. Anecdotal evidence suggests that the number of coin flips is surprisingly large (in the hundreds) even in small classes (a few dozen). In this section we'll analyze this question.

We'll start with calculating the probability that at least one of the students flips a coin m or more times (we may assume m is even, since the number of coin flips is never odd). This probability is

$$1-\left(\sum_{k=1}^{\frac{m}{2}-1}\frac{2}{4^k k}\binom{2k-2}{k-1}\right)^n.$$

Below is a table of values of the probability for a given m and n:

$m\backslash n$	10	15	20	25	30	50	100
10	0.959005	0.9917	0.998319	0.99966	0.999931	1.	1.
100	0.567464	0.715532	0.812913	0.876958	0.919078	0.984861	0.999771
1000	0.225661	0.318608	0.400399	0.472372	0.535705	0.721608	0.922498
10000	0.0769893	0.113232	0.148051	0.181504	0.213642	0.330064	0.551185
100000	0.024947	0.0371861	0.0492716	0.0612053	0.0729893	0.118665	0.223248
1000000	0.00795027	0.0119017	0.0158373	0.0197573	0.0236617	0.0391243	0.0767178

Table 1: Probability that among n people someone flips a coin at least m times before they get the same number of heads as tails.

In the spirit of the birthday paradox, we calculate the following table that represents the number of students n one needs to be 50% sure that one of them will flip the coin at least m times.

m	n
10	3
100	9
1000	28
10000	87
100000	275
1000000	869

Table 2: Number of students n that are needed so that the probability that at least one of them flips a coin at least m times is at least 50%.

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