A pair of identities

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Theorem 1. *Let* $x \in (-1, 1)$, *then*

$$\sum_{k=1}^{\infty} \frac{\mu(k)}{k} \left(-\log\left(1 - x^k\right)\right) = x.$$

Proof.

$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{\mu(k)}{k} \left(-\log\left(1 - x^k\right)\right) &= \sum_{k=1}^{\infty} \frac{\mu(k)}{k} \sum_{i=1}^{\infty} \frac{x^{ki}}{i} \\
&= \sum_{k,i} \frac{\mu(k) x^{ki}}{ki} \\
&= \sum_{m=1}^{\infty} \frac{x^m}{m} \sum_{d|m} \mu(d) \\
&= x.
\end{aligned}$$

The last equation is true because $\sum_{d|n} \mu(d)$ is 1 for $n = 1$ and 0 otherwise. $\square$

Theorem 2. *Let* $x \in (-1, 1)$, *then*

$$\sum_{k=1}^{\infty} \frac{\phi(k)}{k} \left(-\log\left(1 - x^k\right)\right) = \frac{x}{1 - x}.$$

Proof.

$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{\phi(k)}{k} \left(-\log\left(1 - x^k\right)\right) &= \sum_{k=1}^{\infty} \sum_{d|k} \frac{\mu(d)}{k} \left(\frac{k}{d}\right) \sum_{i=1}^{\infty} \frac{x^{ki}}{i} \\
&= \sum_{d=1}^{\infty} \sum_{k=1}^{\infty} \sum_{i=1}^{\infty} \frac{\mu(d) x^{kdi}}{di} \\
&= \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} \frac{x^{km}}{m} \sum_{d|m} \mu(d) \\
&= \sum_{k=1}^{\infty} x^k = \frac{x}{1-x}.
\end{aligned}$$

□