

Bounds on graphs with high girth and high chromatic number

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joint work with Daniel Bath and Zequn Li

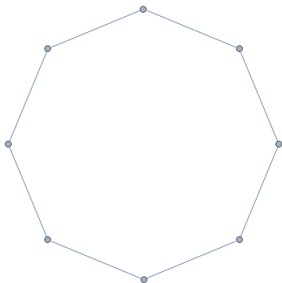
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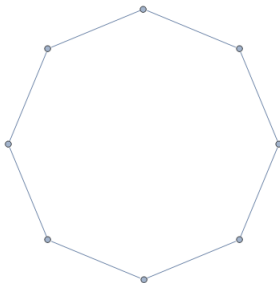
Some Definitions

- **Chromatic Number:** The chromatic number of a graph G is the smallest number of colors needed to color the vertices of G so that no two adjacent vertices share the same color.
- **Cycle:** A cycle graph of length n is an n sided polygon (i.e., a graph with n vertices and n edges where each vertex has degree 2). Example:



More Definitions

- **Independent Set:** An independent set in a graph is a set of vertices no two of which are adjacent.
- **Girth Number :** The girth number of a graph is the length of a shortest cycle contained in the graph. If the graph does not contain any cycles (i.e. it's an acyclic graph), its girth is defined to be infinity. In the following example the girth number is 8:



Examples

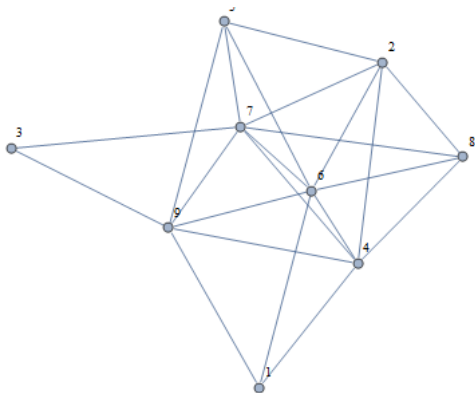


Figure: The maximal independent set consists of vertices $\{8, 1, 3, 5\}$, the chromatic number is at least 5 because the vertices $\{2, 8, 7, 4, 6\}$ form a complete graph. The girth number is 3 because 3 is the smallest cycle.

Erdős, 1959

In the paper “Graph Theory and Probability”, Erdős proved the following:

- Let $h(k, l)$ be the least integer such that every graph with $h(k, l)$ vertices contains either a cycle of k or fewer edges or the graph contains a set of l independent vertices.
- Then

$$h(k, l) > l^{1+\frac{1}{2k}}$$

It's worth noting that this paper was one of the papers where Erdős championed the probabilistic method.

High chromatic number and high girth

- The main consequence of the result mentioned in the previous slide is the following:
- For any integers r and k , there exists a graph $G(r, k)$ that has chromatic number greater than r and girth number greater than k .
- Let $n(r, k)$ be the least integer such that there is a graph $G(r, k)$ of $n(r, k)$ vertices with the property that its chromatic number is $> r$ and its girth number is $> k$. Then

$$n(r, k) = |G(r, k)| < r^{2k+1}.$$

- Suppose $h(k, l) > l^{1+\frac{1}{2k}}$. Let's show that this implies the celebrated result of Erdős:
- Since $h(k, l) > l^{1+\frac{1}{2k}}$, there exists a graph G with $h(k, l) - 1 = n$ vertices such that G has no cycles of length k or less and G has at least l independent vertices.
- Since G has at least l independent vertices, then

$$\chi(G) > \frac{|G|}{l} > l^{\frac{1}{2k}} - \frac{1}{l}.$$

- Therefore $r = l^{\frac{1}{2k}}$ and

$$n(r, k) \leq l^{1+\frac{1}{2k}} = r^{2k+1}.$$

Ideas behind the proof of the lower bound on $h(k, l)$

- Let $G_p(n)$ be the space of random graphs of n vertices where each edge has probability p of being in the graph.
- Let $n = r^{2k+1}$ and let $p = \frac{2r^2}{n}$.
- Give an upperbound to the expected value $\mathbb{E}(C)$ of cycles of length at most k . For instance, it is not hard to show that this expected value is bounded above by

$$\mathbb{E}(C) < \frac{2^{k-1} r^{2k-2}}{3}.$$

- Let A be the set of graphs in $G_p(n)$ that contain at most $f = 2^{k-1} r^{2k-2}$ cycles of length less than g .

Ideas behind the proof of the lower bound on $h(k, l)$

- Now $\mathbb{E}(C) \geq \mathbb{P}_p(\bar{\Omega}_1)f = (1 - \mathbb{P}_p(\Omega_1))f$, and hence $\mathbb{P}_p(\Omega_1) > \frac{2}{3}$.
- Let $l = n/r$ and let Ω_2 be the set of graphs in $G_p(n)$ that do not contain a set of l vertices spanning at most f edges.
- Suppose $G_0 \in \Omega_1 \cap \Omega_2$, then delete an edge from each cycle of length less than k to form graph G . Graph G now has girth at least k . Since $G_0 \in \Omega_2$, then every l -set of vertices spans at least $f + 1$ edges. Also, $G_0 \in \Omega_1$, so the number of edges deleted from G_0 is at most f . So every l -set spans at least one edge, so there are at most $l - 1$ independent points.

Conclusion

- To conclude one need only show $\mathbb{P}(\Omega_1) + \mathbb{P}(\Omega_2) > 1$ (to force an element to be in the intersection of the Ω 's).
- With some clever estimates one can show that in fact $\mathbb{P}(\Omega_1) + \mathbb{P}(\Omega_2) > \frac{4}{3}$.

Theorem

Let $n(r, k)$ be defined as before, then

$$r^{k/2} \ll n(r, k) < r^{2k+1}.$$

Questions

Theorem

Let $n(r, k)$ be defined as before, then

$$r^{k/2} \ll n(r, k) < r^{2k+1}.$$

- Does

$$\lim_{g, k \rightarrow \infty} \frac{\log n(r, k)}{k \log r},$$

exist?

- If the limit exists is it closer to the lower bound (1/2) or to the upper bound (2)?

Some Numerical Work

			Color					
		3	4	5	6	7	8	9
Girth	4	6	8	18	34	50	81	117
	5	8	13	32	93			
	6	9	34	103				
	7	11	47					
	8	12	100					
	9	14						
	10	15						

Table: This table shows the number of vertices the smallest bound we could find that has chromatic number at least r and girth number at least r .

Thank You!