Problems I've invented for math competitions or journals

July 25, 2023

1. ¿Cuántos triángulos isósceles se pueden formar con los vértices de un polígono regular de 21 lados?

   Appeared in the Chihuahua Mathematical Olympiad in some year between 2003 and 2006.

2. Encontrar tres números primos que cumplan con las siguientes condiciones:

$$P_1 < P_2 < P_3.$$
$$2P_2 = P_3 - 25.$$
$$P_1P_3 = 3P_1P_2 + 16.$$

   Encuentre $P_1P_2P_3$.

   Appeared as Problem 4 in the 2006 Chihuahua Mathematical Olympiad.

3. ¿Cuántos números menores a 100000 tienen como dígito más grande un 7 y como segundo dígito más grande un 2? Ejemplos: 2007 cumple, 2777 cumple, 7777 no cumple, 6121 no cumple, 22107 cumple.

   Appeared as Problem 5 in the 2007 Chihuahua Mathematical Olympiad.

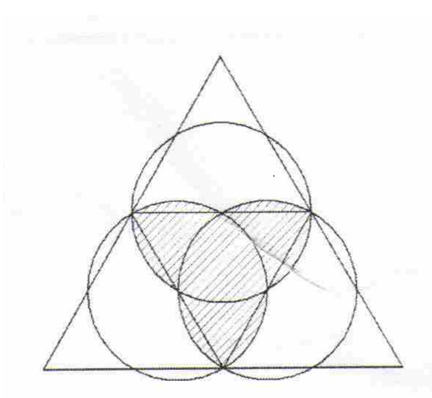
4. Llenamos un tablero de 25×21 al azar con números del 1 al 4. Demuestra que existe un rectángulo tal que la suma de los números en sus orillas es múltiplo de 4.

   Appeared as Problem 6 in the 2007 Chihuahua Mathematical Olympiad. Also appeared (in English) as Problem 10 in the 2010 Thayer prize exam in Dartmouth College. Also appeared in Spanish as Problem 6 in Tzaloa's first issue of 2021.

5. Tienes un triángulo equilátero de lado 4. Se trazan líneas paralelas a los lados que pasen por los puntos medios como se ve en la figura. Luego se hacen círculos cuyo diámetro son los segmentos que unen los puntos medios de cada lado.

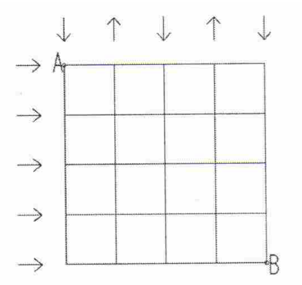
   Si sabemos que el área de un triángulo grande es $4\sqrt{3}$, ¿cuál es el área sombreada?

   Appeared as Problem 3 in the 2008 Cd. Juárez Mathematical Olympiad.



6. El pueblo de Kikelandia tiene 5 calles verticales y 5 calles horizontales. Las calles son todas de un solo sentido con el sentido que indican las flechas. Héctor está en el punto A y quiere llegar al punto B. ¿De cuántas maneras se puede? (Héctor debe seguir el sentido de las calles).

   Appeared as Problem 5 in the 2008 Cd. Juárez Mathematical Olympiad.



7. Dados los números del 1 al 12 escritos en cualquier orden en una mesa redonda se hace la siguiente operación: Cada número observa a sus dos vecinos, si los vecinos son más chicos entonces los números explotan. ¿De cuántas maneras se pueden acomodar los números del 1 al 12 de tal manera que sobrevivan exactamente cuatro después de una operación?

   Appeared as Problem 4 in the 2009 Chihuahua Mathematical Olympiad. Also appeared as Problem 1 in Tzaloa's first issue of 2021.

8. Tenemos los números del 1 al $2k$, ¿de cuántas maneras los puedes emparejar de tal manera que los elementos de cada pareja tienen la misma paridad. Por ejemplo, para $k = 2$: en el conjunto $\{1, 2, 3, 4\}$ un emparejamiento que cumple con las condiciones del problema sería $\{(1, 3), (2, 4)\}$, uno que no cumple es $\{(1, 2), (3, 4)\}$.

   Appeared as Problem 4 in the 2010 Chihuahua Mathematical Olympiad. Also appeared (with a slight modification) as Problem 7 in Tzaloa's first issue of 2021.

9. Un número "chida" es un número divisible por la suma del cuadrado de sus dígitos. Ejemplo, 2010 es "chida" porque $2^2 + 0^2 + 1^2 + 0^2 = 5$ y 5 divide a 2010. Demuestra que existen infinitos números n tales que n y $n + 1$ son "chidas".

   Appeared in a Mexico IMO team selection test around 2010. A version in English appeared as Problem 9 in the 2010 Thayer Prize exam in Dartmouth College.

10. Considera un triángulo rectángulo con catetos a y b e hipotenusa c. Supongamos que el área de este triángulo es la cuarta parte de un cuadrado de lado c. Calcula a/b.

    Appeared in the 2017 Invitational to the Mexican Mathematical Olympiad. This was the first problem I invented, but it took 15 years before it was used somewhere.

11. For a positive integer n, we define $f(n)$ to be the number of 2's that appear (as digits) after writing the numbers $1, 2, \ldots, n$ in their decimal expansion. For example, $f(22) = 6$ because 2 appears once in the numbers 2, 12, 20, 21 and it appears twice in the number 22. Prove that there are finitely many numbers n such that $f(n) = n$.

    Appeared in Mathematical Reflections as Problem O424 in Issue 5 of the year 2017. With the added question of finding an n such that $f(n) = n$, it appeared as Problem 2062 in Math Mag volume 92, issue 1 (2019). I found out this question was posed by Google in the early 2000s and the full sequence of n's such that $f(n) = n$ is sequence A101639 of the Online Encyclopedia of Integer Sequences.

12. ¿Cuántos números n menores a un millón cumplen que $\frac{1}{5}\binom{2n}{n}$ es entero?

    The English version of this problem appeared in Mathematical Reflections Issue 6 of the year 2017. It's Problem O430. An easier variant appeared in the University of Georgia High School Math Tournament as Problem 24 in the 2017 written test.

13. Let A_n be the number of entries in the n-th row of Pascal's triangle that are 1 modulo 3. Let B_n be the number of entries in the n-th row which are 2 modulo 3. Prove that $A_n - B_n$ is a power of 2 for all positive integers n.

    This appeared as Problem O468 in issue 6 of Mathematical Reflections from 2018.

14. Let $a_1 = 0, a_2 = a_3 = a_4 = a_5 = 1, a_6 = a_7 = \cdots = a_{23} = 0, a_{24} = a_{25} = \cdots = a_{119} = 1$, and so on, where the sequence changes from 0 to 1 (or from 1 to 0) at the factorials. For example, since $120 = 5!$, then a_{120} changes from 1 to 0, so $a_{120} = 0$.

    (a) Let
    $$z_n = \frac{1}{n}\sum_{i=1}^{n} a_i.$$
    Does z_n converge? If so, what does it converge to?

    (b) For $n \geq 2$, let
    $$w_n = \frac{1}{\log(n)}\sum_{i=1}^{n} \frac{a_i}{i}.$$
    Does w_n converge? If so, what does it converge to?

    This appeared as Problem 2057 in Math Mag volume 91, issue 5 (2018).

15. Let S be the number of ways of coloring n points in the plane using p colors (p is prime) in such a way that every color is used at least once. Find all n such that S is not a multiple of p^2.

    This appeared as Problem 2070 in Math Mag volume 92, issue 2 (2019).

16. A positive integer n can be written in *factoridic* as $n = \overline{a_k a_{k-1} \ldots a_1}$ for some integers $k, a_1, a_2, \ldots, a_k$ with $k \geq 1$ and $0 \leq a_i \leq i$ for all $i \in \{1, 2, \ldots, k\}$, and
    $$n = a_k \times k! + a_{k-1} \times (k-1)! + \ldots + a_2 \times 2! + a_1 \times 1!.$$
    For example 21 in *factoridic* is written as $\overline{311}$ because $21 = 3 \times 3! + 1 \times 2! + 1 \times 1!$. Suppose that you are given n in factoridic, but they don't delineate which number is a_k, which one is a_{k-1}, et cetera. Sometimes, you can still deduce the the value of n. For example, when $n = \overline{311}$, we have that n must be $3\times3!+1\times2!+1\times1! = 21$. Sometimes you can't deduce the number, as in $n = \overline{10000000000}$, where n could be $1\times11!+0\times(10!+9!+\ldots+1!) = 11!$ or n could be $10\times10!+0\times(9!+8!+7!+\ldots+1!) = 10\times10!$. We say a number like the latter is *ambiguous*. Prove that the number of unambiguous numbers is finite and find the largest unambiguous number (express the number in factoridic).

    This appeared as Problem 2073 in Math Mag volume 92, issue 3 (2019).

17. Evaluate
    $$\sum_{n=0}^{\infty} \frac{\binom{4n}{2n}}{4^{2n}(2n+1)(2n+2)}.$$

    This appeared as Problem 2111 in Math Mag volume 94, issue 1 (2021).

18. Sea $a_0, a_1, a_2, \ldots$ una sucesión de números tales que a_0 es primo y, para $n \geq 1$, $a_n = 2a_{n-1} + 1$. Demuestra que existe un entero $k > 0$ tal que a_k no es primo.

    This appeared as Problem 5 in Tzaloa's first issue of 2021.

19. Consider the grid of n^2 lattice points $\{1, 2, \ldots, n\}^2$. Let $S_1(n)$ be the number of rectangles with corners in the grid (though not necessarily with horizontal and vertical sides) that have area equal to a prime congruent to 1 mod 4. Define $S_3(n)$ similarly using primes congruent to 3 mod 4. Prove that there is a value n_0 such that $S_1(n) > S_3(n)$ for $n \geq n_0$.

    This appeared as Problem 12320 in American Mathematical Monthly volume 129 issue 4 (2022).