# On the maximum number of consecutive integers on which a character is constant

Enrique Treviño

Swarthmore College

AMS-MAA Joint Meetings
January 5, 2012

# Preliminaries

- Let $\chi$ be a non-principal Dirichlet character to the prime modulus $p$.
- Let $H(p)$ be the maximum number of consecutive integers for which $\chi$ is constant.
- Trivially $H(p) \leq p$.
- By the Pólya–Vinogradov inequality, $H(p) \ll p^{1/2} \log p$.
- By the Burgess inequality, $H(p) \ll_\varepsilon p^{1/4+\varepsilon}$.

# Explicit Estimates

Let $\chi$ be a non-principal Dirichlet character to the prime modulus $p$. Let $H(p)$ be the maximum number of consecutive integers for which $\chi$ is constant.

**Theorem (Burgess, 1963)**

$H(p) = O(p^{1/4} \log p)$.

**Theorem (McGown, 2011)**

$H(p) < \left\{ \frac{\pi e \sqrt{6}}{3} + o(1) \right\} p^{1/4} \log p$.

*Furthermore,*

$$H(p) \leq \begin{cases} 7.06 p^{1/4} \log p, & \text{for } p \geq 5 \cdot 10^{18}, \\ 7 p^{1/4} \log p, & \text{for } p \geq 5 \cdot 10^{55}. \end{cases}$$

In a conference in 1973, Norton made the following claim without proof:

**Claim**

*$H(p) \leq 2.5p^{1/4} \log p$ for $p > e^{15} \approx 3.27 \times 10^6$ and $H(p) < 4.1p^{1/4} \log p$ for all odd $p$.*

I was able to prove the claim (and a little more), improving on the theorem of McGown.

# Main Theorem

## Theorem (T)

*Let $\chi$ be a non-principal Dirichlet character to the prime modulus p. Let H(p) be the maximum number of consecutive integers for which $\chi$ is constant, then*

$$H(p) < \left\{ \frac{\pi}{2}\sqrt{\frac{e}{3}} + o(1) \right\} p^{1/4} \log p.$$

*Furthermore,*

$$H(p) \leq \begin{cases} 3.64 p^{1/4} \log p, & \text{for all odd } p, \\ 1.55 p^{1/4} \log p, & \text{for } p \geq 2.5 \cdot 10^9. \end{cases}$$

# Main ingredient

The main ingredient in the proof comes from estimating

$$S_\chi(h,w) = \sum_{m=1}^{p} \left| \sum_{l=0}^{h-1} \chi(m+l) \right|^{2w}.$$

- Burgess showed that $S_\chi(h,w) < (4w)^{w+1} p h^w + 2wp^{1/2}h^{2w}$.
- McGown improved it to $S_\chi(h,w) < \frac{1}{4}(4w)^w p h^w + (2w-1)p^{1/2}h^{2w}$.
- For quadratic characters, Booker showed $S_\chi(h,w) < \frac{(2w)!}{2^w w!} p h^w + (2w-1)p^{1/2}h^{2w}$.
- I showed that Booker's inequality holds for all characters.

# Lower bound Lemma

## Lemma

*Let $h$ and $w$ be positive integers. Let $\chi$ be a non-principal Dirichlet character to the prime modulus $p$ which is constant on $(N, N+H]$ and such that*

$$4h \leq H \leq \left(\frac{h}{2}\right)^{2/3} p^{1/3}.$$

*Let $X := H/h$, then $X \geq 4$ and*

$$S_\chi(h, w) \geq \left(\frac{3}{\pi^2}\right) X^2 h^{2w+1} g(X) = AH^2 h^{2w-1} g(X),$$

*where $A = \frac{3}{\pi^2}$, and*

$$g(X) = 1 - \left(\frac{13}{12AX} + \frac{1}{4AX^2}\right).$$

# Laborious Proof of Lemma

Let $a$ and $b$ be integers satisfying $1 \leq a \leq \left\lfloor \frac{2H}{h} \right\rfloor$ and

$$\left| a\frac{N}{p} - b \right| \leq \frac{1}{\left\lfloor \frac{2H}{h} \right\rfloor + 1} \leq \frac{h}{2H}.$$

Now define $I(q, t)$ to be the real interval:

$$I(q,t) := \left( \frac{N + pt}{q}, \frac{N + H + pt}{q} \right],$$

for integers $0 \leq t < q \leq X$ and $\gcd(at + b, q) = 1$.

Given

$$I(q,t) := \left(\frac{N+pt}{q}, \frac{N+H+pt}{q}\right],$$

we have

- $\chi$ is constant inside the interval $I(q,t)$ since if $m \in I(q,t)$, then

  $$\chi(q)\chi(m) = \chi(qm) = \chi(qm - pt) = \chi(N+i),$$

  where $i \in (0, H]$.
- The $I(q,t)$ are disjoint (for this you need to use the restriction on $H$ and on $a$).
- $I(q,t) \subset (0,p)$.

Since the $I(q,t)$ are disjoint and they are contained in $(0,p)$, we have

$$S_\chi(h,w) = \sum_{m=0}^{p-1} \left| \sum_{l=0}^{h-1} \chi(m+l) \right|^{2w} \geq \sum_{q,t} \sum_{m \in I(q,t)} \left| \sum_{l=0}^{h-1} \chi(m+l) \right|^{2w}$$

$$\geq h^{2w} \sum_{q,t} \left( \frac{H}{q} - h \right) = h^{2w+1} \sum_{q \leq X} \sum_{\substack{0 \leq t < q \\ \gcd(at+b,q)=1}} \left( \frac{X}{q} - 1 \right).$$

Recall that $X = H/h$. Evaluating this last sum yields our lemma. The main difference between the technique Burgess and McGown use is that they have $X = H/(2h)$ and then they evaluate the last sum with 1 instead of $\left(\frac{X}{q} - 1\right)$.

# Finishing the Proof of the Main Theorem

- Combining the upper and lower bounds on $S_\chi(h,w)$ we have

$$AH^2h^{2w-1}g(X) \leq S_\chi(h,w) < \frac{(2w)!}{2^w w!}ph^w + (2w-1)p^{1/2}h^{2w}.$$

- Optimizing for $h$ and $w$ asymptotically, we find the asymptotic in our theorem and we also prove that $H(p) < 1.55p^{1/4}\log p$ for $p \geq 10^{64}$.
- Since $H < \left(\frac{h}{2}\right)^{2/3} p^{1/3}$, we have to juggle a bit and we find that we are constrained to $p \geq 2.5 \times 10^9$.
- To cover the gaps between $2.5 \times 10^9$ and $10^{64}$ we pick specific $h$'s and $w$'s and check for intervals as depicted in the table on the following slide.

# Table

| $w$ | $h$ | $p$ | $w$ | $h$ | $p$ | $w$ | $h$ | $p$ |
|---|---|---|---|---|---|---|---|---|
| 6 | 26 | $[2.5 \cdot 10^9, 10^{10}]$ | 6 | 28 | $[10^{10}, 4 \cdot 10^{10}]$ | 7 | 28 | $[4 \cdot 10^{10}, 10^{11}]$ |
| 7 | 32 | $[10^{11}, 10^{12}]$ | 7 | 37 | $[10^{12}, 10^{13}]$ | 8 | 41 | $[10^{13}, 10^{14}]$ |
| 8 | 44 | $[10^{14}, 10^{15}]$ | 9 | 45 | $[10^{15}, 10^{16}]$ | 9 | 51 | $[10^{16}, 10^{17}]$ |
| 9 | 59 | $[10^{17}, 10^{18}]$ | 10 | 62 | $[10^{18}, 10^{19}]$ | 11 | 63 | $[10^{19}, 10^{20}]$ |
| 11 | 71 | $[10^{20}, 10^{21}]$ | 12 | 72 | $[10^{21}, 10^{23}]$ | 13 | 79 | $[10^{23}, 10^{25}]$ |
| 15 | 82 | $[10^{25}, 10^{27}]$ | 15 | 96 | $[10^{27}, 10^{29}]$ | 17 | 97 | $[10^{29}, 10^{31}]$ |
| 18 | 105 | $[10^{31}, 10^{33}]$ | 18 | 119 | $[10^{33}, 10^{35}]$ | 19 | 127 | $[10^{35}, 10^{37}]$ |
| 20 | 135 | $[10^{37}, 10^{39}]$ | 20 | 149 | $[10^{39}, 10^{41}]$ | 22 | 150 | $[10^{41}, 10^{43}]$ |
| 23 | 158 | $[10^{43}, 10^{46}]$ | 25 | 166 | $[10^{46}, 10^{49}]$ | 27 | 174 | $[10^{49}, 10^{52}]$ |
| 29 | 183 | $[10^{52}, 10^{55}]$ | 31 | 191 | $[10^{55}, 10^{58}]$ | 33 | 200 | $[10^{58}, 10^{62}]$ |
| 33 | 215 | $[10^{62}, 10^{64}]$ | | | | | | |

Table: As an example on how to read the table: when $w = 10$ and $h = 62$, then the constant 1.55 works for all $p \in [10^{18}, 10^{19}]$. It is also worth noting that the inequality $1.55p^{1/4} \log p < h^{2/3}p^{1/3}$ is also verified for each choice of $w$ and $h$.

# For all $p$

To get a bound for all $p$ we use the following theorem (established with elementary methods):

**Theorem (Brauer)**

$$H(p) < \sqrt{2p} + 2.$$

Using this, one can show that $H(p) < 3.64p^{1/4}\log p$ whenever $p < 3 \times 10^6$. Using the techniques from before one can show that for $p \geq 3 \times 10^6$, $H(p) < 3.64p^{1/4}\log p$. One of the obstacles preventing us from getting a lower number is the restriction $H < \left(\frac{h}{2}\right)^{2/3} p^{1/3}$.

# Thank you!