Summer Research Projects for First Year Students

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Lake Forest College

Joint Math Meetings
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LAKE FOREST COLLEGE

Richter Scholar Program

- The Richter Scholar Summer Research Program provides students with the opportunity to conduct independent, individual research with Lake Forest College faculty early in their academic careers (have to be first year students).
- Students work one-on-one with a faculty member, doing independent research in one of a wide variety of fields. At the end of the program students present their work either via a poster or a presentation at the Richter Scholar Symposium.

Projects

- What's so rational about the alphabet?
  Marina Rawlings and Kevin Kupiec.
  Summer 2014, three week project.
- Finding perfect polynomials mod 2.
  Ugur Caner Cengiz.
  Summer 2014, 10 week project.
- On Tupper's self-referential formula.
  Margaret Fortman.
  Summer 2015, 4 week project.

Projects Continued

- First-world solutions to First-year problems.
  Robert Mecham
  Summer 2015, 10 week project.
- Beatty sequences and the prime race.
  Noel Orwothwun
  Summer 2016, 4 week project.

Marina and Kevin

Background

- "Walking on Real Numbers" by Aragón, Bailey, Borwein and Borwein.
- Consider a number in base 4. For example π. In base 4,

  $$\pi = 3.021003331222202020112203002031030 1\ldots,$$

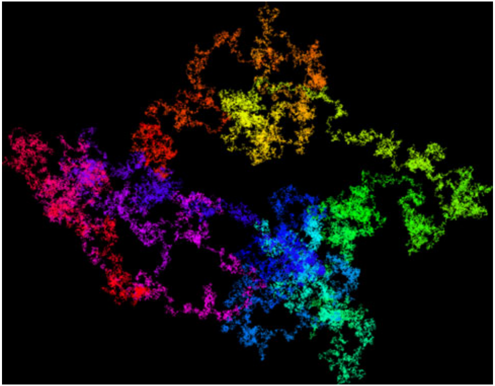
  because

  $$\pi = 3 + 0\left(\frac{1}{4}\right) + 2\left(\frac{1}{4^2}\right) + 1\left(\frac{1}{4^3}\right) + \ldots$$

- We start at the origin in the Cartesian plane. We move a unit to the right whenever we hit a digit 0, we move a unit up whenever we hit a digit 1, we move a unit left whenever we hit a digit 2 and we move a unit down if we hit a digit 3.

Walking on π

Walking on the first 100 billion digits of π reveals the following picture:



Easier Example

Walking on the number 419636198 which can be rewritten in base 4 as

$$121000302033212_4.$$

Inspiration

10490122716774994374866192805654486016 17567358491560876166848380843144358447
25287555162924702775955557045371567931 30587832477297720217708181879659063736
57674879814228013285920278610192581409 57135748704712290267465151312805954195
39975042020613803738223389597133391954

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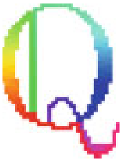
16122269626942909129404900662735492142 29880755725468512353395718465191353017
34881431401750453996944547935301206438 33272670970079330526292030350920973600
45095545613659664932507839146477284016 23856513742952945308961226815274887561
56580761624107880751845994219387748355



Project

- For each letter of the alphabet, find a rational that satisfies that if you random walk through that rational, you get the letter of the alphabet.
- Build a computer program that can use the information above to figure out the rational that works for a particular phrase.

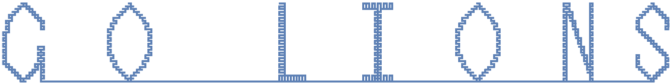
Example: Numerator has 1070 digits

1031270085734598033814281924937329589675258434769478812573135352158342
9734157060576763003770728598527412303639118539265497462213421611731239
8749319847832333845799243665089916978093486379310564201451994274541739
8953940586141894932146026156360719214931871061507258715181956852269121
9109895646083562881016483738809878340281009181021038155735970996648843
8471474717313366876572670913397016409447998311805692146671867213603632
1126262321177603810475253927903868351994488290373906359133113241317924
4945771424500973077693331292609175892646439643224196163465328649476952
1535175531768833235782917377898222942819455515362709152983758219208834
9877327302878345677194442920231504255669051754074394761978755410923039
1582467585547935387697487732609417103169633294967806775635539236390770
1318010651907355105052654682456985593431664673242291857199749221305728
5223816147551411296123746791577122964940704506940301444947330312017174
2905634429731387933525055373972958171653391211137048718726709337879181
0068554754516627356653060589017916209209043142250590321326303055341430
5978681971202029431

/

1813612909395326365821731915974724074694418250409562751208229013067788
6799244153370929609065333968674392478053056257764294858769874160777326
4981437577125453150781424590687736087049390561768296318040590082331216
8861432833119411535835071124340395479459512785017586145742960663209294
9609512044538622340627589478971957142917146838612458341577198250170427
1175188859314828831659170048959195742465352707516932323821024590224481
2744912006813158678063511811898789132287064169569488764792809447208998
1620159888784747000564340828377043115520617513421789001320318761912136
3789326173819309629426902794997676751067653270637748886499713085761540
7921849308480839304298636148148923334161898996047666156249017462165176
0389861974636035118614227785697397948930749744269457874528545234951247
8156430016534013054670700327808098797756500897367805666055305120854325
8976592499261606352090557072671782859712164708955424388208352292258691
3171946989606962685102833965666861076762057232891615209379928498982210
8146960592326046449523170784841062548760777675447195976875966562215219
930239750921739304955

Picture

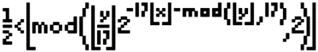


Margaret

Tupper's self-referential formula

$$\frac{1}{2} < \left\lfloor \operatorname{mod}\left(\left\lfloor \frac{y}{17} \right\rfloor 2^{-17\lfloor x \rfloor - \operatorname{mod}(\lfloor y \rfloor, 17)}, 2 \right) \right\rfloor$$

Graph of the above equation for $0 \le x \le 105$ and $k \le y \le k + 17$ is



for k equal to
4858450636189713423582095962494202044581400587983244549483093085061934704708809
9284506447698655243648499972470249151191104116057391774078569197543265718554420572
1044573588368182982375413963433822519945219165128434833290513119319995350241375876
5239264874613394906870130562295813219481113685339535565290850023875092856892694555
9742815463865107300491067230589335860525440966643512653493636439571255656959368151
8433485760526694016125126695142155053955451915378545752575659074054015792900176596
796548006442782913148854825991472124850635268663047630O

Project

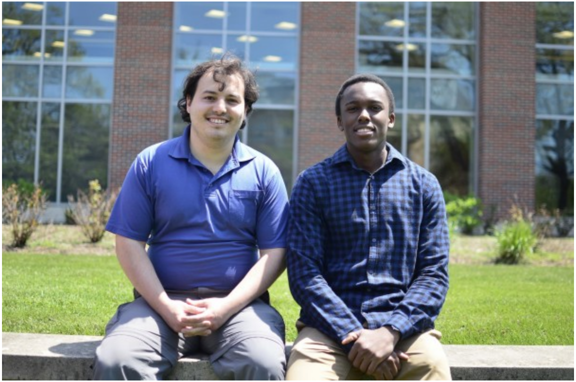
For a given sentence, find the integer k such that the graph of Tupper's formula looks like that sentence for $0 \leq x \leq 105$ and $k \leq y \leq k + 17$.

Example

Plot of Tupper's formula for $0 \le x \le 105$ and $k \le y \le k + 17$ when k is

2739647459486629045536465740775152818439155040716788201151421285564313
8886055687869724088333511646748444038613911787291657027640710254540775
2388071116970815545870091930990081870477840782406472469008475204294244
0337447584237103246084720952767916330378661088431056533806844699692204
1720114559659824254161882825703573629449288644913587553313598874377831
4638820738501573396643297951978710457949104145903290655519730079558563
8197866416615171834312293074216566395180181235145885225190087168490221
8932943401798006209950838627654250114 7901952

Noel



Beatty Sequences

- Given a positive irrational number θ, the Beatty sequence associated to θ is the sequence of integers:

$$\lfloor \theta \rfloor, \lfloor 2\theta \rfloor, \ldots$$

- If two positive irrational numbers α, β satisfy $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, then we say that the Beatty sequences for α and β are complementary since their union is $\mathbb{N}$ and the intersection is empty.
- For the rest of this section, let A be the Beatty sequence associated to α and B the one associated to β. Then

$$A \cup B = \mathbb{N} \quad A \cap B = \emptyset.$$

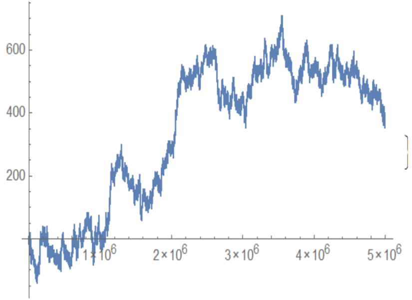


Figure: Difference for $\alpha = \sqrt{2}$ up to 5000000

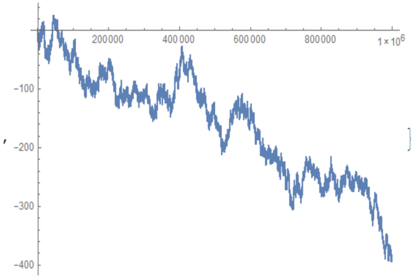


Figure: Difference for $\alpha = \sqrt{3}$ up to 1000000

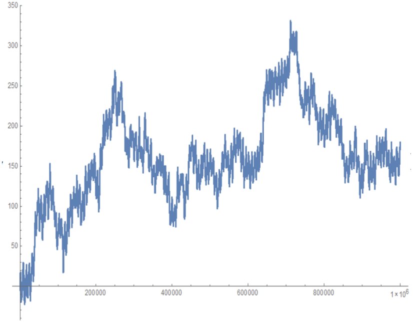
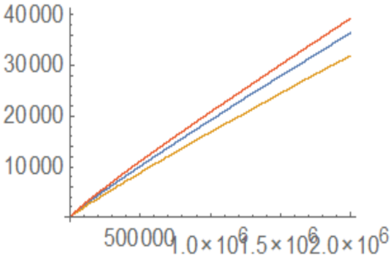


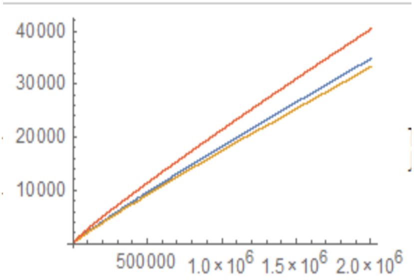
Figure: Difference for $\alpha = \pi$ up to 1000000

Bias between consecutive primes

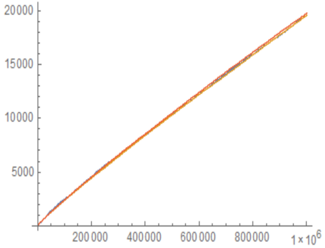
- Soundararajan and Lemke-Oliver recently (this year!) discovered that there's a repulsion among the last digit of consecutive primes. For example if a prime ends in the digit 1, the next prime is less likely to end with digit 1 than randomness suggests.
- This phenomenon happens in any modulus. We'll focus on mod 3 where primes (other than 3) have two possibilities, i.e., 1 mod 3 and 2 mod 3.
- Is there such a bias with Beatty sequences?

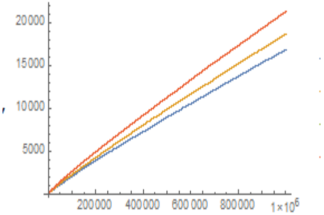
In the following pictures, the red color represents when the consecutive primes land on different Beatty sequences, the blue color represents when they both land on *A*, and the yellow color when they both land on B.
The first graph is for $\alpha = \sqrt{2}$, the second is for $\alpha = \sqrt{3}$ for numbers up to 2000000.





These ones are for $\alpha = \pi$, and $\alpha = e$ for numbers up to 1000000.





Robert

First year studies

- At Lake Forest College every incoming first year student gets assigned a "first-year-studies" course.
- Students rank their choices.
- The administration then decides where to send students depending on several factors:
  1. All classes should have roughly the same number of students.
  2. Some classes have an ACT/SAT threshold.
  3. Some classes do not allow athletes (because the classes do many field trips).
- Can we build a computer program that does this for the administration?

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Answer

Yes.

Caner

Polynomials Mod 2

- A polynomial mod 2 is one of the form

$$a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0,$$

where $a_i \in \{0, 1\}$.

- For example

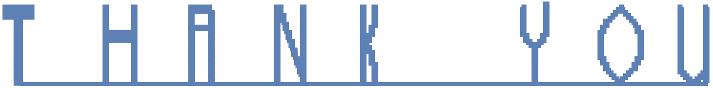
$$x^2 + 1 = x^2 + 2x + 1 = (x+1)^2.$$

Perfect Polynomials Mod 2

- Let $\sigma(P)$ be the sum of the divisors of a polynomial P in mod 2.
- A polynomial is said to be perfect mod 2 if $\sigma(P) = P$.
- $x^2 + x = x(x+1)$, so
$$\sigma(x^2 + x) = 1 + x + (x+1) + x^2 + x = x^2 + x.$$
So it is perfect.
- $$\sigma(x^2 + 1) = 1 + (1 + x) + (1 + x^2) = 1 + x + x^2.$$

Degree	Factorization into Irreducibles
5	$T(T+1)^2(T^2+T+1)$
	$T^2(T+1)(T^2+T+1)$
11	$T(T+1)^2(T^2+T+1)^2(T^4+T+1)$
	$T^2(T+1)(T^2+T+1)^2(T^4+T+1)$
	$T^3(T+1)^4(T^4+T^3+1)$
	$T^4(T+1)^3(T^4+T^3+T^2+T+1)$
15	$T^3(T+1)^6(T^3+T+1)(T^3+T^2+1)$
	$T^6(T+1)^3(T^3+T+1)(T^3+T^2+1)$
16	$T^4(T+1)^4(T^4+T^3+1)(T^4+T^3+T^2+T+1)$
20	$T^4(T+1)^6(T^3+T+1)(T^3+T^2+1)(T^4+T^3+T^2+T+1)$
	$T^6(T+1)^4(T^3+T+1)(T^3+T^2+1)(T^4+T^3+1)$

2604492038498829664517035139638426870137917317601914921681571572073092
8695649142399363187609280194294167878123410715564434829372484954369221
6688674071161566295182034925025124500569335817787439872314183988703371
664977034810026841939388184409144925414508598131377588818477056



2448384190858685068867681259343425875183951488272013453075082431177592 0962050203904557003580654935217327434523590196726040466969501801090605
5664172807097345932129875992132606806463166306220789443630207743373850 7810762263639386835401550161949045105588754258631414471096220189442622
1665427118990256510546429000993755547847039756192941247474274244110787 8660478187344039509845541294428633493975708789645581674249072107946910
2765408044063942823929158894523762592434026978546624093591607103258197 4450638724117458467731275518440553284813684834902522605636902712794432
6983720681160278275882713626711046114853618093566074424105013908484306 7404052939064793858614545855382152074663980612417168491376157942144755
1424213635574427532612628956112547281010460985326871870253127060031406 9020366919487757012852917356230205755317365507728854738429219183094869
3695972821843598905429765758338285668885349223378577986330045089333457 9166101811912895915931607121962558396750697854916244499605957383922242
2087768224286810034216057699362187013656283147052173522331783042174874 4140094804118333840723288129692077185337544126726009078432030025686771
5239809488805445016460285761824177350795554860429732609262786222124676 600894826038486209391328014641316052644422022

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3819479337739529167329838484906512027721012110801860682548053881724174 2920102672707614237693622767847895409616985207326646402685739366166208
4840626414213587363092837135985370821617952963431759840705122747193031 7627890537469797349469887395417031631058274040351838367832032124521106
7194847614828901800156489387408054011676810457486834475529542970498905 7255432583059392620413461586291555620523049770255248031290472335172139
2412944217588056092569134224444251794977065504945068111268913987049510 5866348693513857856730829398652779264556390283916331450215630875473236
7905955243200201225507949865086092749749275242859745654420665019701251 9591247855104320397246547508188151581997031018121793852983572059720856
2703336970436575992101116604676953965084910204328651037047629049824332 4556534083393752340059349194724035480526710224113881565618356297919255
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2535713206663022363291361908461665812355382922360345773026535557687152 4820925610319529593771464824025674303842751783960198988339742665429620
7718565411907925539753987771468284452619059692514367146906203118121384 335388589276651377014630572190734743924675145