Partitioning powers into sets of equal sum

Enrique Treviño

LAKE FOREST COLLEGE

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Motivating Puzzle

Consider the following puzzle submitted by Dean Ballard to the Riddler column on the FiveThirtyEight website:

King Auric adored his most prized possession: a set of perfect spheres of solid gold. There was one of each size, with diameters of 1 centimeter, 2 centimeters, 3 centimeters, and so on. Their brilliant beauty brought joy to his heart. After many years, he felt the time had finally come to pass the golden spheres down to the next generation – his three children. He decided it was best to give each child precisely one-third of the total gold by weight, but he had a difficult time determining just how to do that. After some trial and error, he managed to divide his spheres into three groups of equal weight. He was further amused when he realized that his collection contained the minimum number of spheres needed for this division. How many golden spheres did King Auric have?

Translation and a variant

What is the smallest positive integer n such that $1^3, 2^3, \ldots, n^3$ can be partitioned into three sets with equal sum?

Question

For what positive integers n can the set $\{1^3, 2^3, \ldots, n^3\}$ be partitioned into three sets with equal sum?

Thinking it through

A partition of the set $\{n, n-1, n-2, n-3, n-4, n-5\}$ into three sets of equal sum is

$$\begin{array}{c|c|c} n & n-1 & n-2 \\ n-5 & n-4 & n-3 \end{array}$$

If we square them, the quadratic and linear terms match, but not the constant terms. To make the constant terms match, we can include more numbers and permute.

$$\begin{array}{c|c|c} n & n-1 & n-2 \\ n-5 & n-4 & n-3 \\ n-8 & n-6 & n-7 \\ n-9 & n-11 & n-10 \\ n-13 & n-14 & n-12 \\ n-16 & n-15 & n-17 \end{array}$$

Nice Theorem

Theorem

Any set of 54 consecutive cubes can be partitioned into three sets of equal sum.

Generalization:

Theorem

Any $2m^k$ consecutive k-th powers can be partitioned into m sets of equal sum.

Connecting to the Math Literature

The Prouhet-Tarry-Escott (PTE) problem asks to find integers N, and a_{ij} for $i \in \{1, 2, \ldots, N\}, j \in \{1, 2, \ldots, m\}$ such that

$$\begin{aligned}
\sum_{j=1}^{N} a_{1j} = \sum_{j=1}^{N} a_{2j} &= \cdots = \sum_{j=1}^{N} a_{mj} \\
\sum_{j=1}^{N} a_{1j}^2 = \sum_{j=1}^{N} a_{2j}^2 &= \cdots = \sum_{j=1}^{N} a_{mj}^2 \\
&\vdots \\
\sum_{j=1}^{N} a_{1j}^k = \sum_{j=1}^{N} a_{2j}^k &= \cdots = \sum_{j=1}^{N} a_{mj}^k.
\end{aligned} \tag{1}$$

The trivial PTE solutions are those for which there is an i and a $j \neq i$ for which the sets $\{a_{i1}, a_{i2}, \ldots, a_{iN}\}, \{a_{j1}, a_{j2}, \ldots, a_{jN}\}$ are the same.

What we know of PTE

Let $P(k, m)$ be the smallest positive integer N such that there is a nontrivial solution to PTE.

- $P(k, m) \geq k + 1$
- (Borwein-Ingalls) $P(k, m) = k + 1$ for $s \leq 10$ or $s = 12$ and $m = 2$, or for $k \in \{2, 3, 5\}$ for any m
- (Wright)

$$P(k, m) \leq \begin{cases} \frac{k^2+4}{2} & \text{for } m = 2 \\ \frac{k^2+3}{2} & \text{for odd } k \\ \frac{k^2+k+2}{2} & \text{otherwise} \end{cases}.$$

- Prouhet gave an explicit construction showing $P(k, m) \leq m^k$.

Our theorem in the PTE framework

Theorem

There is an explicit construction showing $P(k, m) \leq 2m^{k-1}$.

Back to our puzzle

- We need $1^3 + 2^3 + \ldots + n^3$ to be a multiple of 3. Therefore $n \equiv 0, 2 \mod 3$.
- We know that if it can be partitioned for n it can also be for $n + 54$.

Searching for partitions

n	Partition
23	$\{3,6,10,13,18,19,21\}, \{1,4,7,8,12,16,20,22\}, \{2,5,9,11,14,15,17,23\}$
26	$\{4,14,19,24,26\}, \{2,3,5,11,15,16,18,22,25\}, \{1,6,7,8,9,10,12,13,17,20,21,23\}$
27	$\{11,12,21,25,27\}, \{7,13,14,15,17,18,22,26\}, \{1,2,3,4,5,6,8,9,10,16,19,20,23,24\}$
29	$\{7,12,14,19,24,25,28\}, \{2,6,8,17,20,23,26,27\}, \{1,3,4,5,9,10,11,13,15,16,18,21,22,29\}$
30	$\{4,7,8,16,19,25,26,30\}, \{3,5,11,14,17,20,21,23,24,27\}, \{1,2,6,9,10,12,13,15,18,22,28,29\}$
32	$\{16,22,25,31,32\}, \{2,4,5,8,11,12,17,18,19,20,23,29,30\},$ $\{1,3,6,7,9,10,13,14,15,21,24,26,27,28\}$
33	$\{4,13,16,21,24,26,28,33\}, \{1,3,6,7,10,18,20,25,27,29,31\},$ $\{2,5,8,9,11,12,14,15,17,19,22,23,30,32\}$
35	$\{7,17,24,25,28,32,35\}, \{11,18,19,20,22,29,33,34\},$ $\{1,2,3,4,5,6,8,9,10,12,13,14,15,16,21,23,26,27,30,31\}$
36	$\{5,7,10,14,22,29,31,33,35\}, \{1,6,12,15,16,17,18,20,24,26,27,28,36\},$ $\{2,3,4,8,9,11,13,19,21,23,25,30,32,34\}$
38	$\{5,17,21,24,29,32,35,38\}, \{1,6,9,10,12,13,14,15,20,31,33,36,37\},$ $\{2,3,4,7,8,11,16,18,19,22,23,25,26,27,28,30,34\}$
39	$\{6,22,25,27,36,37,39\}, \{2,3,4,5,8,11,12,13,16,23,26,29,30,32,33,35\},$ $\{1,7,9,10,14,15,17,18,19,20,21,24,28,31,34,38\}$
41	$\{2,5,18,26,27,32,35,39,41\}, \{10,13,20,23,24,28,31,34,38,40\},$ $\{1,3,4,6,7,8,9,11,12,14,15,16,17,19,21,22,25,29,30,33,36,37\}$
42	$\{2,8,9,20,24,26,35,38,39,42\}, \{3,4,6,7,11,12,14,15,19,21,25,36,37,40,41\},$ $\{1,5,10,13,16,17,18,22,23,27,28,29,30,31,32,33,34\}$
44	$\{1,2,9,20,28,31,36,38,43,44\}, \{4,5,8,10,11,13,15,22,25,26,27,29,32,39,40,42\},$ $\{3,6,7,12,14,16,17,18,19,21,23,24,30,33,34,35,37,41\}$
45	$\{5,10,11,19,24,27,28,29,32,34,36,40,44\}, \{4,9,14,15,22,23,26,30,31,37,39,41,42\},$ $\{1,2,3,6,7,8,12,13,16,17,18,20,21,25,33,35,38,43,45\}$
47	$\{16,24,25,28,34,38,43,45,47\}, \{5,7,10,20,21,31,35,36,37,40,42,46\},$ $\{1,2,3,4,6,8,9,11,12,13,14,15,17,18,19,22,23,26,27,29,30,32,33,39,41,44\}$
48	$\{8,10,12,19,22,24,25,27,32,36,37,39,45,48\}, \{2,3,5,6,9,11,18,20,23,29,40,41,42,46,47\},$ $\{1,4,7,13,14,15,16,17,21,26,28,30,31,33,34,35,38,43,44\}$

Searching for partitions 2

n	Partition
50	{6, 12, 16, 28, 33, 36, 41, 42, 43, 45, 49}, {1, 2, 9, 18, 19, 20, 22, 25, 27, 29, 30, 31, 34, 35, 37, 39, 46, 47}, {3, 4, 5, 7, 8, 10, 11, 13, 14, 15, 17, 21, 23, 24, 26, 32, 38, 40, 44, 48, 50}
51	{2, 8, 11, 15, 20, 26, 32, 37, 42, 44, 45, 47, 49}, {3, 14, 16, 22, 25, 28, 30, 31, 34, 35, 38, 43, 50, 51}, {1, 4, 5, 6, 7, 9, 10, 12, 13, 17, 18, 19, 21, 23, 24, 27, 29, 33, 36, 39, 40, 41, 46, 48}
53	{4, 6, 13, 17, 18, 21, 30, 32, 46, 47, 49, 51, 53}, {3, 8, 24, 25, 27, 31, 36, 38, 42, 44, 45, 48, 52}, {1, 2, 5, 7, 9, 10, 11, 12, 14, 15, 16, 19, 20, 22, 23, 26, 28, 29, 33, 34, 35, 37, 39, 40, 41, 43, 50}
54	{17, 22, 38, 39, 47, 48, 49, 51, 52}, {4, 5, 18, 24, 26, 33, 36, 40, 42, 43, 45, 53, 54}, {1, 2, 3, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 19, 20, 21, 23, 25, 27, 28, 29, 30, 31, 32, 34, 35, 37, 41, 44, 46, 50}
56	{3, 5, 8, 11, 16, 23, 26, 27, 30, 31, 32, 38, 43, 44, 45, 47, 52, 53}, {2, 6, 7, 9, 10, 12, 13, 17, 18, 19, 20, 21, 22, 24, 25, 29, 33, 35, 40, 41, 46, 48, 54, 55}
57	{7, 31, 40, 52, 53, 55, 56, 57}, {2, 3, 12, 13, 17, 18, 20, 22, 23, 25, 26, 27, 29, 32, 36, 38, 39, 41, 42, 43, 47, 48, 54}, {1, 4, 5, 6, 8, 9, 10, 11, 14, 15, 16, 19, 21, 24, 28, 30, 33, 34, 35, 37, 44, 45, 46, 49, 50, 51}
59	{1, 2, 6, 9, 10, 18, 23, 27, 37, 38, 43, 46, 55, 56, 57, 58}, {3, 4, 5, 11, 13, 15, 17, 20, 30, 33, 34, 35, 36, 44, 48, 50, 53, 54, 59}, {7, 8, 12, 14, 16, 19, 21, 22, 24, 25, 26, 28, 29, 31, 32, 39, 40, 41, 42, 45, 47, 49, 51, 52}
60	{14, 16, 23, 26, 30, 31, 32, 35, 38, 40, 42, 43, 44, 47, 50, 56, 57}, {1, 2, 3, 5, 7, 11, 13, 15, 19, 21, 29, 33, 36, 37, 45, 49, 53, 55, 58, 60}, {4, 6, 8, 9, 10, 12, 17, 18, 20, 22, 24, 25, 27, 28, 34, 39, 41, 46, 48, 51, 52, 54, 59}
62	{2, 4, 7, 9, 11, 21, 30, 31, 48, 50, 51, 57, 58, 60, 62}, {1, 3, 8, 10, 15, 18, 19, 22, 29, 33, 39, 42, 47, 49, 52, 53, 54, 56, 59}, {5, 6, 12, 13, 14, 16, 17, 20, 23, 24, 25, 26, 27, 28, 32, 34, 35, 36, 37, 38, 40, 41, 43, 44, 45, 46, 55, 61}
63	{1, 4, 5, 8, 15, 23, 24, 29, 35, 42, 46, 51, 53, 55, 58, 60, 61}, {7, 10, 11, 12, 14, 17, 18, 21, 28, 33, 34, 37, 38, 39, 40, 44, 47, 48, 49, 52, 59, 62}, {2, 3, 6, 9, 13, 16, 19, 20, 22, 25, 26, 27, 30, 31, 32, 36, 41, 43, 45, 50, 54, 56, 57, 63}

Searching for partitions 3

n	Partition
65	$\{1, 7, 8, 18, 22, 25, 27, 32, 38, 40, 48, 50, 52, 55, 63, 64, 65\}$ $\{2, 4, 6, 11, 14, 15, 16, 19, 21, 28, 30, 34, 37, 39, 41, 43, 49, 56, 58, 59, 61, 62\}$, $\{3, 5, 9, 10, 12, 13, 17, 20, 23, 24, 26, 29, 31, 33, 35, 36, 42, 44, 45, 46, 47, 51, 53, 54, 57, 60\}$
66	$\{3, 7, 9, 13, 17, 18, 23, 31, 34, 41, 46, 49, 53, 55, 58, 61, 62, 65\}$, $\{2, 4, 6, 8, 14, 20, 24, 29, 32, 36, 37, 38, 40, 47, 48, 54, 59, 60, 63, 66\}$, $\{1, 5, 10, 11, 12, 15, 16, 19, 21, 22, 25, 26, 27, 28, 30, 33, 35, 39, 42, 43, 44, 45, 50, 51, 52, 56, 57, 64\}$
68	$\{1, 2, 6, 7, 13, 33, 36, 37, 41, 48, 54, 55, 57, 58, 62, 64, 68\}$, $\{3, 5, 11, 16, 17, 20, 22, 23, 25, 26, 27, 32, 38, 42, 47, 49, 50, 51, 52, 59, 60, 63, 66\}$, $\{4, 8, 9, 10, 12, 14, 15, 18, 19, 21, 24, 28, 29, 30, 31, 34, 35, 39, 40, 43, 44, 45, 46, 53, 56, 61, 65, 67\}$
69	$\{1, 6, 7, 28, 33, 38, 39, 40, 55, 57, 60, 63, 65, 67, 68\}$, $\{3, 4, 9, 14, 15, 20, 21, 22, 24, 25, 30, 31, 32, 35, 45, 49, 58, 61, 62, 64, 66, 69\}$, $\{2, 5, 8, 10, 11, 12, 13, 16, 17, 18, 19, 23, 26, 27, 29, 34, 36, 37, 41, 42, 43, 44, 46,$ $47, 48, 50, 51, 52, 53, 54, 56, 59\}$
71	$\{2, 8, 14, 18, 20, 22, 31, 35, 37, 39, 43, 46, 47, 51, 53, 55, 57, 61, 62, 66, 67\}$, $\{4, 5, 9, 10, 11, 13, 17, 24, 26, 28, 30, 33, 38, 41, 44, 49, 52, 56, 58, 63, 64, 70, 71\}$, $\{1, 3, 6, 7, 12, 15, 16, 19, 21, 23, 25, 27, 29, 32, 34, 36, 40, 42, 45, 48, 50, 54, 59, 60, 65, 68, 69\}$
72	$\{1, 2, 3, 5, 10, 17, 25, 26, 34, 38, 46, 49, 52, 54, 55, 56, 59, 61, 62, 68, 69\}$, $\{4, 7, 12, 14, 15, 20, 23, 24, 28, 29, 33, 40, 41, 48, 50, 51, 53, 57, 58, 60, 66, 67, 70\}$, $\{6, 8, 9, 11, 13, 16, 18, 19, 21, 22, 27, 30, 31, 32, 35, 36, 37, 39, 42, 43, 44, 45, 47, 63, 64, 65, 71, 72\}$
74	$\{1, 2, 4, 8, 9, 18, 21, 22, 27, 29, 46, 47, 50, 52, 56, 57, 64, 67, 70, 72, 73\}$, $\{5, 6, 10, 11, 12, 13, 14, 17, 32, 33, 34, 43, 44, 48, 49, 51, 53, 62, 63, 65, 66, 68, 74\}$, $\{3, 7, 15, 16, 19, 20, 23, 24, 25, 26, 28, 30, 31, 35, 36, 37, 38, 39, 40, 41, 42, 45, 54, 55, 58, 59, 60, 61, 69, 71\}$
75	$\{1, 2, 3, 6, 7, 12, 13, 17, 25, 26, 40, 43, 48, 57, 59, 60, 64, 71, 72, 73, 75\}$, $\{4, 9, 19, 20, 23, 24, 28, 34, 37, 38, 46, 47, 49, 50, 51, 54, 55, 56, 61, 63, 65, 69, 70\}$, $\{5, 8, 10, 11, 14, 15, 16, 18, 21, 22, 27, 29, 30, 31, 32, 33, 35, 36, 39, 41, 42, 44, 45, 52, 53, 58, 62, 66, 67, 68, 74\}$
78	$\{1, 6, 11, 12, 20, 31, 40, 42, 43, 45, 46, 54, 62, 67, 69, 72, 74, 76, 78\}$, $\{2, 3, 4, 5, 7, 9, 22, 33, 37, 38, 48, 49, 51, 52, 55, 59, 63, 64, 71, 73, 75, 77\}$, $\{8, 10, 13, 14, 15, 16, 17, 18, 19, 21, 23, 24, 25, 26, 27, 28, 29, 30, 32, 34, 35, 36, 39, 41, 44, 47,$ $50, 53, 56, 57, 58, 60, 61, 65, 66, 68, 70\}$

Answer to the Puzzle and to some variants

Theorem

The $\{1^3, 2^3, \ldots, n^3\}$ *can be partitioned into three sets of equal sum if and only if* $n = 23$ *or* $n \geq 26$ *with* $n \equiv 0, 2 \mod 3$.

Theorem

1. *The first n cubes can be partitioned into two sets of equal sum if and only if* $n \geq 12$ *and* $n \equiv 0, 3 \mod 4$.
2. *The first n squares can be partitioned into two sets of equal sum if and only if* $n \geq 7$ *and* $n \equiv 0, 3 \mod 4$.
3. *The first n squares can be partitioned into three sets of equal sum if and only if* $n \geq 18$ *and* $n \equiv 0, 4, 8 \mod 9$.

Thank you

Thank You