Prime gaps: a breakthrough in number theory

Enrique Treviño

Faculty Discussion Group, September 16, 2014

LAKE FOREST COLLEGE

Introduction

Consider the sequence

11 12 13 14 15 16 17 18 19 20

- How many even numbers does it have?
- It has 5.
- How many square numbers?
- It has 1.
- How many prime numbers?
- It has 4.

Introduction

Consider the sequence

11 12 13 14 15 16 17 18 19 20

- How many even numbers does it have?
- It has 5.
- How many square numbers?
- It has 1.
- How many prime numbers?
- It has 4.

Introduction

Consider the sequence

11 12 13 14 15 16 17 18 19 20

- How many even numbers does it have?
- It has 5.
- How many square numbers?
- It has 1.
- How many prime numbers?
- It has 4.

Introduction

Consider the sequence

11 12 13 14 15 16 17 18 19 20

- How many even numbers does it have?
- It has 5.
- How many square numbers?
- It has 1.
- How many prime numbers?
- It has 4.

Consider the sequence from 1001 to 2000:

- How many even numbers does it have?
- It has 500.
- How many square numbers?
- It has 13.
- How many prime numbers?
- It has 135.

Consider the sequence from 1001 to 2000:

- How many even numbers does it have?
- It has 500.
- How many square numbers?
- It has 13.
- How many prime numbers?
- It has 135.

Consider the sequence from 1001 to 2000:

- How many even numbers does it have?
- It has 500.
- How many square numbers?
- It has 13.
- How many prime numbers?
- It has 135.

Consider the sequence from 1001 to 2000:

- How many even numbers does it have?
- It has 500.
- How many square numbers?
- It has 13.
- How many prime numbers?
- It has 135.

Consider the sequence from 1000001 and 2000000:

- How many even numbers does it have?
- It has 500000.
- How many square numbers?
- It has 414.
- How many prime numbers?
- It has 70435.

Consider the sequence from 1000001 and 2000000:

- How many even numbers does it have?
- It has 500000.
- How many square numbers?
- It has 414.
- How many prime numbers?
- It has 70435.

Consider the sequence from 1000001 and 2000000:

- How many even numbers does it have?
- It has 500000.
- How many square numbers?
- It has 414.
- How many prime numbers?
- It has 70435.

Consider the sequence from 1000001 and 2000000:

- How many even numbers does it have?
- It has 500000.
- How many square numbers?
- It has 414.
- How many prime numbers?
- It has 70435.

Introduction

Consider the sequence

11 12 13 14 15 16 17 18 19 20

- What is the largest and smallest gap between two consecutive even numbers?
- The answer for both is 2.
- What is the largest and smallest gap between two consecutive square numbers?
- This question is not applicable.
- What is the largest and smallest gap between two consecutive square numbers?
- The largest is 4 and the smallest is 2.

Introduction

Consider the sequence

11 12 13 14 15 16 17 18 19 20

- What is the largest and smallest gap between two consecutive even numbers?
- The answer for both is 2.
- What is the largest and smallest gap between two consecutive square numbers?
- This question is not applicable.
- What is the largest and smallest gap between two consecutive square numbers?
- The largest is 4 and the smallest is 2.

Introduction

Consider the sequence

11 12 13 14 15 16 17 18 19 20

- What is the largest and smallest gap between two consecutive even numbers?
- The answer for both is 2.
- What is the largest and smallest gap between two consecutive square numbers?
- This question is not applicable.
- What is the largest and smallest gap between two consecutive square numbers?
- The largest is 4 and the smallest is 2.

Introduction

Consider the sequence

11 12 13 14 15 16 17 18 19 20

- What is the largest and smallest gap between two consecutive even numbers?
- The answer for both is 2.
- What is the largest and smallest gap between two consecutive square numbers?
- This question is not applicable.
- What is the largest and smallest gap between two consecutive square numbers?
- The largest is 4 and the smallest is 2.

Consider the sequence from 1001 to 2000:

- What is the largest and smallest gap between two consecutive even numbers?
- The answer for both is 2.
- What is the largest and smallest gap between two consecutive square numbers?
- The largest is 87 and the smallest is 65.
- What is the largest and smallest gap between two consecutive square numbers?
- The largest is 34 and the smallest is 2.

Consider the sequence from 1001 to 2000:

- What is the largest and smallest gap between two consecutive even numbers?
- The answer for both is 2.
- What is the largest and smallest gap between two consecutive square numbers?
- The largest is 87 and the smallest is 65.
- What is the largest and smallest gap between two consecutive square numbers?
- The largest is 34 and the smallest is 2.

Consider the sequence from 1001 to 2000:

- What is the largest and smallest gap between two consecutive even numbers?
- The answer for both is 2.
- What is the largest and smallest gap between two consecutive square numbers?
- The largest is 87 and the smallest is 65.
- What is the largest and smallest gap between two consecutive square numbers?
- The largest is 34 and the smallest is 2.

Consider the sequence from 1001 to 2000:

- What is the largest and smallest gap between two consecutive even numbers?
- The answer for both is 2.
- What is the largest and smallest gap between two consecutive square numbers?
- The largest is 87 and the smallest is 65.
- What is the largest and smallest gap between two consecutive square numbers?
- The largest is 34 and the smallest is 2.

Consider the sequence from 1000001 and 2000000:

- What is the largest and smallest gap between two consecutive even numbers?
- The answer for both is 2.
- What is the largest and smallest gap between two consecutive square numbers?
- The largest is 2827 and the smallest is 2001.
- What is the largest and smallest gap between two consecutive square numbers?
- The largest is 132 and the smallest is 2.

Consider the sequence from 1000001 and 2000000:

- What is the largest and smallest gap between two consecutive even numbers?
- The answer for both is 2.
- What is the largest and smallest gap between two consecutive square numbers?
- The largest is 2827 and the smallest is 2001.
- What is the largest and smallest gap between two consecutive square numbers?
- The largest is 132 and the smallest is 2.

Consider the sequence from 1000001 and 2000000:

- What is the largest and smallest gap between two consecutive even numbers?
- The answer for both is 2.
- What is the largest and smallest gap between two consecutive square numbers?
- The largest is 2827 and the smallest is 2001.
- What is the largest and smallest gap between two consecutive square numbers?
- The largest is 132 and the smallest is 2.

Consider the sequence from 1000001 and 2000000:

- What is the largest and smallest gap between two consecutive even numbers?
- The answer for both is 2.
- What is the largest and smallest gap between two consecutive square numbers?
- The largest is 2827 and the smallest is 2001.
- What is the largest and smallest gap between two consecutive square numbers?
- The largest is 132 and the smallest is 2.

Range	# Evens	# Squares	# Primes
11-20	5	1	4
1001-2000	500	13	135
1000001-2000000	500000	414	70435

	Evens		Squares			Primes		
R	#	Gaps	#	S Gap	B Gap	#	S Gap	B Gap
10	5	2	1	NA	NA	4	2	4
10^3	500	2	13	65	87	135	2	34
10^6	$5 \cdot 10^5$	2	414	2001	2827	70435	2	132
x	$\frac{x}{2}$	2	$(\sqrt{2}-1)\sqrt{x}$	$2\sqrt{x}$	$2\sqrt{2x}$	$\frac{x}{\log x}$	2	$(\log x)^2$

Prime Gaps

Range	#	Small Gap	Average Gap	Large Gap
10	4	2	2	4
1000	135	2	7.33	34
1000000	70435	2	14.20	132
x	$\frac{x}{\log x}$	2	$\log x$	$(\log x)^2$

Twin Prime Conjectures

Conjecture (Twin Prime)

There are infinitely many primes p such that $p+2$ is also prime.

Let's make a slightly stronger conjecture that we can analyze as a function of a variable x:

Conjecture

Let x be a large integer. There is a prime p between $x+1$ and $2x$ such that $p+2$ is also prime. That is, in the graph from the previous slide, the smallest gap when the range is x will be 2.

Theorem

For x a large integer, there are two primes p and q between $x+1$ and $2x$ such that $|q-p| \leq \log x$.

Results on short gaps

Let $S(x)$ be the smallest gap between two primes between $x+1$ and $2x$. Then

- $\frac{S(x)}{\log x} \leq 1$.
- $\frac{S(x)}{\log x} \leq 1 - c$, for infinitely many x and a fixed $c > 0$ (Erdős, 1940).
- $\frac{S(x)}{\log x} \leq \frac{1}{2}$ for infinitely many x (Bombieri-Vinogradov, 1966).
- $\frac{S(x)}{\log x} \leq \frac{1}{2}e^{-\gamma} = 0.2807\ldots$ for infinitely many x (Maier 1988).
- $\frac{S(x)}{\log x} \leq 1/4$ for infinitely many x (Maier).
- $\liminf_{x\to\infty} \frac{S(x)}{\log x} = 0$. (Goldston-Pintz-Yildirim, 2005)

Results on short gaps

Let $S(x)$ be the smallest gap between two primes between $x+1$ and $2x$. Then

- $\frac{S(x)}{\log x} \leq 1$.
- $\frac{S(x)}{\log x} \leq 1 - c$, for infinitely many x and a fixed $c > 0$ (Erdős, 1940).
- $\frac{S(x)}{\log x} \leq \frac{1}{2}$ for infinitely many x (Bombieri-Vinogradov, 1966).
- $\frac{S(x)}{\log x} \leq \frac{1}{2} e^{-\gamma} = 0.2807\ldots$ for infinitely many x (Maier 1988).
- $\frac{S(x)}{\log x} \leq 1/4$ for infinitely many x (Maier).
- $\liminf_{x \to \infty} \frac{S(x)}{\log x} = 0$. (Goldston-Pintz-Yildirim, 2005)

Results on short gaps

Let $S(x)$ be the smallest gap between two primes between $x+1$ and $2x$. Then

- $\frac{S(x)}{\log x} \leq 1$.
- $\frac{S(x)}{\log x} \leq 1-c$, for infinitely many x and a fixed $c > 0$ (Erdős, 1940).
- $\frac{S(x)}{\log x} \leq \frac{1}{2}$ for infinitely many x (Bombieri-Vinogradov, 1966).
- $\frac{S(x)}{\log x} \leq \frac{1}{2}e^{-\gamma} = 0.2807\ldots$ for infinitely many x (Maier 1988).
- $\frac{S(x)}{\log x} \leq 1/4$ for infinitely many x (Maier).
- $\liminf_{x\to\infty} \frac{S(x)}{\log x} = 0$. (Goldston-Pintz-Yildirim, 2005)

Results on short gaps

Let $S(x)$ be the smallest gap between two primes between $x+1$ and $2x$. Then

- $\dfrac{S(x)}{\log x} \leq 1.$
- $\dfrac{S(x)}{\log x} \leq 1 - c$, for infinitely many x and a fixed $c > 0$ (Erdős, 1940).
- $\dfrac{S(x)}{\log x} \leq \dfrac{1}{2}$ for infinitely many x (Bombieri-Vinogradov, 1966).
- $\dfrac{S(x)}{\log x} \leq \dfrac{1}{2}e^{-\gamma} = 0.2807\ldots$ for infinitely many x (Maier 1988).
- $\dfrac{S(x)}{\log x} \leq 1/4$ for infinitely many x (Maier).
- $\liminf\limits_{x\to\infty} \dfrac{S(x)}{\log x} = 0.$ (Goldston-Pintz-Yildirim, 2005)

Results on short gaps

Let $S(x)$ be the smallest gap between two primes between $x+1$ and $2x$. Then

- $\frac{S(x)}{\log x} \leq 1$.
- $\frac{S(x)}{\log x} \leq 1 - c$, for infinitely many x and a fixed $c > 0$ (Erdős, 1940).
- $\frac{S(x)}{\log x} \leq \frac{1}{2}$ for infinitely many x (Bombieri-Vinogradov, 1966).
- $\frac{S(x)}{\log x} \leq \frac{1}{2} e^{-\gamma} = 0.2807\ldots$ for infinitely many x (Maier 1988).
- $\frac{S(x)}{\log x} \leq 1/4$ for infinitely many x (Maier).
- $\liminf_{x \to \infty} \frac{S(x)}{\log x} = 0$. (Goldston-Pintz-Yildirim, 2005)

Goldston-Pintz-Yildirim

Theorem (GPY)

Let $\epsilon > 0$, then for infinitely many x,

$$\liminf_{x\to\infty} \frac{S(x)}{(\log x)^{1/2+\epsilon}} = 0.$$

Furthermore, if the Elliott-Halberstam conjecture is true,

$$S(x) \leq 16,$$

for infinitely many x.

Zhang's Theorem

In May 2013, Yitang Zhang, a lecturer (at the time) at the University of New Hampshire managed to break the bounded barrier when he proved the following:

Theorem (Zhang)

For infinitely many x,

$$S(x) \leq 70000000.$$

Polymath8 Progress

Date	ϖ or (ϖ, δ)	k_0	H
Aug 10 2005		6 [EH]	16 [EH] ([Goldston-Pintz-Yildirim])
May 14 2013	1/1,168 (Zhang)	3,500,000 (Zhang)	70,000,000 (Zhang)
May 21			63,374,611 (Lewko)
May 28			59,874,594 (Trudgian)
May 30			59,470,640 (Morrison) 58,885,998? (Tao) 59,093,364 (Morrison) 57,554,086 (Morrison)
May 31		2,947,442 (Morrison) 2,618,607 (Morrison)	48,112,378 (Morrison) 42,543,038 (Morrison) 42,342,946 (Morrison)
Jun 1			42,342,924 (Tao)
Jun 2		866,605 (Morrison)	13,008,612 (Morrison)
			4,982,086 (Morrison)

Polymath8 Progress

Jun 3	1/1,040? (v08ltu)	341,640 (Morrison)	4,982,086 (Morrison) 4,802,222 (Morrison)
Jun 4	1/224?? (v08ltu) 1/240?? (v08ltu)		4,801,744 (Sutherland) 4,788,240 (Sutherland)
Jun 5		34,429? (Paldi/v08ltu) 34,429? (Tao/v08ltu/Harcos)	4,725,021 (Elsholtz) 4,717,560 (Sutherland) 397,110? (Sutherland) 4,656,298 (Sutherland) 389,922 (Sutherland) 388,310 (Sutherland) 388,284 (Castryck) 388,248 (Sutherland) 388,188 (Sutherland) 387,982 (Castryck) 387,974 (Castryck)
Jun 6	~~(1/488,3/9272)~~ (Pintz) ~~1/552~~ (Pintz, Tao)	~~60,000*~~ (Pintz) ~~52,295*~~ (Peake) ~~11,123~~ (Tao)	387,960 (Angelveit) 387,910 (Sutherland) 387,904 (Angeltveit) 387,814 (Sutherland) 387,766 (Sutherland) 387,754 (Sutherland)

Polymath8 Progress

			[illegible] (Sutherland) ~~768,534*~~ (Pintz)
Jun 7	~~(1/538, 1/660)~~ (v08ltu) ~~(1/538, 31/20444)~~ (v08ltu) ~~(1/942, 19/27004)~~ (v08ltu) $828\varpi + 172\delta < 1$ (v08ltu/Green)	~~11,018~~ (Tao) ~~10,721~~ (v08ltu) ~~10,719~~ (v08ltu) ~~25,111~~ (v08ltu) 26,024? (vo8ltu)	~~113,520~~? (Angeltveit) ~~109,314~~? (Angeltveit/Sutherland) ~~707,328*~~ (Sutherland) ~~108,990~~ (Sutherland) ~~113,462*~~ (Sutherland) ~~112,302*~~ (Sutherland) ~~112,272*~~ (Sutherland) ~~116,386*~~ (Sun) ~~108,978~~ (Sutherland) ~~108,634~~ (Sutherland) ~~108,632~~ (Castryck) ~~108,600~~ (Sutherland) ~~108,570~~ (Castryck) ~~108,556~~ (Sutherland) ~~108,550~~ (xfxie) ~~275,424~~ (Sutherland) ~~108,540~~ (Sutherland) ~~275,418~~ (Sutherland) ~~275,404~~ (Sutherland)

Polymath8 Progress

Jun 24	$(134+\frac{2}{3})\varpi + 28\delta \le 1$? (v08ltu) $140\varpi + 32\delta < 1$? (Tao) ~~1/88?? (Tao)~~ ~~1/74?? (Tao)~~	1,268? (v08ltu)	10,206? (Engelsma)
Jun 25	$116\varpi + 30\delta < 1$? (Fouvry-Kowalski-Michel-Nelson/Tao)	1,346? (Hannes) ~~502?? (Trevino)~~ 1,007? (Hannes)	10,876? (Engelsma) ~~3,612?? (Engelsma)~~ 7,860? (Engelsma)
Jun 26	$116\varpi + 25.5\delta < 1$? (Nielsen) $(112+\frac{4}{7})\varpi + (27+\frac{6}{7})\delta < 1$? (Tao)	962? (Hannes)	7,470? (Engelsma)
Jun 27	$108\varpi + 30\delta < 1$? (Tao)	902? (Hannes)	6,966? (Engelsma)
Jul 1	$(93+\frac{1}{3})\varpi + (26+\frac{2}{3})\delta < 1$? (Tao)	873? (Hannes) ~~872? (xfxie)~~	6,712? (Sutherland) ~~6,696? (Engelsma)~~
Jul 5	$(93+\frac{1}{3})\varpi + (26+\frac{2}{3})\delta < 1$ (Tao)	720 (xfxie/Harcos)	5,414 (Engelsma)
Jul 10	7/600? (Tao)		
[illegible]	$(85+\frac{5}{?})\varpi + (25+\frac{5}{?})\delta < 1$?		

Polymath8 Results

Polymath8 was able to improve Zhang's result to:

$$S(x) \leq 4680$$

for infinitely many x.

Maynard and Polymath8b

In November of 2013, Maynard, a postdoc at U. Montreal came out with a different proof of the bounded small gaps. A proof that does not require an improvement on Bombieri-Vinogradov:

Theorem (Maynard)

For infinitely many x:

$$S(x) \leq 600.$$

Furthermore if the Elliott-Halberstam conjecture is true, then for infinitely many x

$$S(x) \leq 12.$$

Polymath8b

Polymath8 joined Maynard and they are improving his result. The latest results (updated September 15, 2014) are:

Theorem (Polymath8b)

For infinitely many x,

$$S(x) \leq 246.$$

Furthermore if EH is true, then for infinitely many x

$$S(x) \leq 6.$$

There seems to be some slight room for improvement to lower 246, but the second result (a bound of 6) is staying put. The famous sieve parity barrier is preventing any improvement there.

Thank you!