Egyptian equations and modern mathematics

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joint work with
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LFC Faculty Discussion
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Scribed in 16th century B.C. by Ahmose.

It’s a copy of a papyrus from the 19th century B.C.

Contains a table of fractions $2/n$ written as sum of unit fractions.

Contains some algebra problems. ALGEBRA!

Contains some geometry. In particular, one can deduce that Egyptians approximated $\pi$ to be

$$\frac{256}{81} \approx 3.1605.$$
Egyptian multiplication

Let’s multiply $33 \times 47$ the Egyptian way.

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
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</thead>
<tbody>
<tr>
<td>1</td>
<td>47</td>
</tr>
<tr>
<td>2</td>
<td>94</td>
</tr>
<tr>
<td>4</td>
<td>188</td>
</tr>
<tr>
<td>8</td>
<td>376</td>
</tr>
<tr>
<td>16</td>
<td>752</td>
</tr>
<tr>
<td>32</td>
<td>1504</td>
</tr>
</tbody>
</table>

Since $1 + 32 = 33$ we have

$$33 \times 47 = 1504 + 47 = 1551.$$
Let’s multiply $47 \times 33$ the Egyptian way.

\[
\begin{array}{c|c}
1 & 33 \\
2 & 66 \\
4 & 132 \\
8 & 264 \\
16 & 528 \\
32 & 1056 \\
\end{array}
\]

Since $1 + 2 + 4 + 8 + 32 = 47$ we have

\[
47 \times 33 = 33 + 66 + 132 + 264 + 1056 = 1551.
\]
Suppose we want to evaluate $33^{47} \mod 100$

\[
\begin{align*}
33^1 & = 33 \\
33^2 & = 89 \\
33^4 & = 21 \\
33^8 & = 41 \\
33^{16} & = 81 \\
33^{32} & = 61
\end{align*}
\]

Since $1 + 2 + 4 + 8 + 32 = 47$ we have

\[
33^{47} = 33^1 \cdot 33^2 \cdot 33^4 \cdot 33^8 \cdot 33^{32} \equiv 33 \cdot 89 \cdot 21 \cdot 41 \cdot 61 \equiv 77 \mod 100.
\]
Table of $2/n$

“Rules” followed:
- Express $2/n$ as the sum of at most 4 unit fractions.
- Have all fractions with denominator $\leq 1000$ (and don’t repeat fractions).

<table>
<thead>
<tr>
<th>$n$</th>
<th>Rhind Mathematical Papyrus</th>
<th>Greedy Algorithm</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>$\frac{1}{2} + \frac{1}{6}$</td>
<td>$\frac{1}{2} + \frac{1}{6}$</td>
</tr>
<tr>
<td>5</td>
<td>$\frac{1}{3} + \frac{1}{15}$</td>
<td>$\frac{1}{3} + \frac{1}{15}$</td>
</tr>
<tr>
<td>7</td>
<td>$\frac{1}{4} + \frac{1}{28}$</td>
<td>$\frac{1}{4} + \frac{1}{28}$</td>
</tr>
<tr>
<td>9</td>
<td>$\frac{1}{6} + \frac{1}{18}$</td>
<td>$\frac{1}{5} + \frac{1}{45}$</td>
</tr>
<tr>
<td>11</td>
<td>$\frac{1}{6} + \frac{1}{66}$</td>
<td>$\frac{1}{6} + \frac{1}{66}$</td>
</tr>
<tr>
<td>13</td>
<td>$\frac{1}{8} + \frac{1}{52} + \frac{1}{104}$</td>
<td>$\frac{1}{7} + \frac{1}{91}$</td>
</tr>
<tr>
<td>15</td>
<td>$\frac{1}{10} + \frac{1}{30}$</td>
<td>$\frac{1}{8} + \frac{1}{120}$</td>
</tr>
<tr>
<td>67</td>
<td>$\frac{1}{40} + \frac{1}{335} + \frac{1}{536}$</td>
<td>$\frac{34}{34} + \frac{1}{2278}$</td>
</tr>
<tr>
<td>89</td>
<td>$\frac{1}{60} + \frac{1}{356} + \frac{1}{534} + \frac{1}{890}$</td>
<td>$\frac{45}{45} + \frac{1}{4005}$</td>
</tr>
</tbody>
</table>
Egyptian Fractions

Any positive rational $a/n$ can be written as the sum of positive unit fractions

$$\frac{a}{n} = \frac{1}{m_1} + \frac{1}{m_2} + \cdots + \frac{1}{m_k}.$$

The above is an example of an Egyptian fraction decomposition of length $k$.

Here’s an example of length 5:

$$\frac{867}{5309} = \frac{1}{7} + \frac{1}{49} + \frac{1}{23650} + \frac{1}{683592739} + \frac{1}{4205691294638106350}$$
Mathematical Questions from Egyptian Fractions

- How many terms do you need?
- How big are the denominators?
- What are the best choices for denominators?
- How long does the “greedy algorithm” take?
- How many representations of $a/b$ as a sum of $n$ Egyptian fractions are there?

Mathematicians working on these questions include Butler, Croot, Erdős, Graham, Konyagin, Yokota
Erdős–Straus conjecture

Conjecture

There exist positive integers $m_1, m_2, m_3$ such that

$$\frac{4}{n} = \frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3}.$$

For example:

$$\frac{4}{8675309} = \frac{1}{2168828} + \frac{1}{6271751022618} + \frac{1}{59002291334558621370338268}$$

- It has been verified for $n \leq 10^{14}$.
- The set of exceptions has density 0. (Vaughan 1970)
Averaging

Since we can’t prove it, let’s average!

**Theorem (Elsholtz–Tao (2013))**

Let 

\[ f(n) = \# \left\{ (m_1, m_2, m_3) \in \mathbb{N}^3 : \frac{4}{n} = \frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} \right\}, \]

then

\[ x \log^2 x \ll \sum_{p \leq x} f(p) \ll x \log^2 x \log \log x. \]
Length 2 and 3

\[ A_k(n) = \# \left\{ a \in \mathbb{N} : \frac{a}{n} = \frac{1}{m_1} + \frac{1}{m_2} + \cdots + \frac{1}{m_k} \right\}. \]

**Theorem (Croot, Dobbs, Friedlander, Hetzel, Pappalardi (2000))**

For any \( \varepsilon > 0 \),

\[ x \log^4 x \ll \sum_{n \leq x} A_2(n) \ll x \log^4 x. \]

**Theorem (Luca–Pappalardi (2019))**

\[ x \log^3 x \ll \sum_{p \leq x} A_3(p) \ll x \log^5 x. \]
Theorem (Croot, Dobbs, Friedlander, Hetzel, Pappalardi (2000))

For any \( \varepsilon > 0 \),
\[
A_3(n) \ll n^{\frac{1}{2} + \varepsilon}.
\]

Theorem (Banderier, Gómez Ruiz, Luca, Pappalardi, Treviño)

Let \( h(n) = C / \log \log n \), where
\[
C = \frac{2 \log(48) \log(\log(6983776800))}{\log(6983776800)} \approx 1.066.
\]
Then
\[
A_3(n) \leq 10n^{\frac{1}{2} + \frac{13}{4} h(n) \log n}.
\]

Corollary

For \( n \geq 10^{10^{23}} \),
\[
A_3(n) \leq \frac{1}{100} n^{\frac{1}{2} + \frac{1}{15}}.
\]
\[ F(n) = \# \left\{ (a, m_1, m_2, m_3) \in \mathbb{N}^4 : \frac{a}{n} = \frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} \right\}. \]

**Theorem (Banderier, Gómez Ruiz, Luca, Pappalardi, Treviño)**

Let \( \varepsilon > 0 \), then

\[ F(n) \ll n^{5/6 + \varepsilon}. \]

This implies that for large enough \( n \), \( F(n) < n \). This suggests the question, what is the largest \( n \) such that \( F(n) \geq n \).

The first values for which \( F(n) < n \) are:

\[ F(8821) = 8590, \quad F(11161) = 10270, \quad F(11941) = 10120. \]
Theorem (Banderier, Gómez Ruiz, Luca, Pappalardi, Treviño)

For $n \geq 10^{10^{23}}$.

$$F(n) \leq \frac{1}{10} n$$
Lemma (Luca, Pappalardi)

Consider an Egyptian fraction decomposition of the irreducible fraction $a/n$:

$$\frac{a}{n} = \frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} \quad \text{with } \gcd(a, n) = 1 \quad (1)$$

Then there exist integers $D_1, D_2, D_3, v_1, v_2, v_3$ with

(i) $\text{lcm}(D_1, D_2, D_3) | n$ and $\gcd(D_1, D_2, D_3) = 1$;

(ii) $av_1v_2v_3 | D_1v_1 + D_2v_2 + D_3v_3$ and $\gcd(v_i, D_jv_j) = 1$ when $i \neq j$,

and the denominators of the Egyptian fractions are given by

$$m_i = \frac{n(D_1v_1 + D_2v_2 + D_3v_3)}{aD_iv_i} \quad (2)$$

Conversely, if conditions (i)–(ii) are fulfilled, then the $m_i$’s defined via (2) are integers, and denominators of $k$ unit fractions summing to $a/n$. 
Thank you!