Egyptian equations and modern mathematics

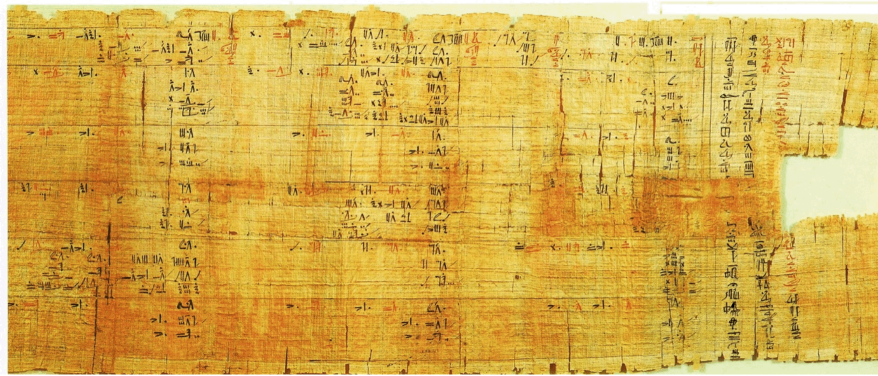
Enrique Treviño

joint work with
C. Banderier, C. A. Gómez Ruiz, F. Luca, F. Pappalardi

LAKE FOREST COLLEGE

LFC Faculty Discussion
January 29, 2020

Rhind Mathematical Papyrus



Rhind Mathematical Papyrus

- Scribed in 16th century B.C. by Ahmose.
- It's a copy of a papyrus from the 19th century B.C.
- Contains a table of fractions $2/n$ written as sum of unit fractions.
- Contains some algebra problems. ALGEBRA!
- Contains some geometry. In particular, one can deduce that Egyptians approximated π to be

$$\frac{256}{81} \approx 3.1605.$$

Egyptian multiplication

Let's multiply 33×47 the Egyptian way.

1	47
2	94
4	188
8	376
16	752
32	1504

Since $1 + 32 = 33$ we have

$$33 \times 47 = 1504 + 47 = 1551.$$

Egyptian multiplication

Let's multiply 47×33 the Egyptian way.

1	33
2	66
4	132
8	264
16	528
32	1056

Since $1 + 2 + 4 + 8 + 32 = 47$ we have

$$47 \times 33 = 33 + 66 + 132 + 264 + 1056 = 1551.$$

Connection to Modern Mathematics

Suppose we want to evaluate $33^{47} \bmod 100$

33^1	33
33^2	89
33^4	21
33^8	41
33^{16}	81
33^{32}	61

Since $1 + 2 + 4 + 8 + 32 = 47$ we have

$$33^{47} = 33^1 \cdot 33^2 \cdot 33^4 \cdot 33^8 \cdot 33^{32} \equiv 33 \cdot 89 \cdot 21 \cdot 41 \cdot 61 \equiv 77 \bmod 100.$$

Table of $2/n$

"Rules" followed:

- Express $2/n$ as the sum of at most 4 unit fractions.
- Have all fractions with denominator ≤ 1000 (and don't repeat fractions).

n	Rhind Mathematical Papyrus	Greedy Algorithm
3	$\frac{1}{2}+\frac{1}{6}$	$\frac{1}{2}+\frac{1}{6}$
5	$\frac{1}{3}+\frac{1}{15}$	$\frac{1}{3}+\frac{1}{15}$
7	$\frac{1}{4}+\frac{1}{28}$	$\frac{1}{4}+\frac{1}{28}$
9	$\frac{1}{6}+\frac{1}{18}$	$\frac{1}{5}+\frac{1}{45}$
11	$\frac{1}{6}+\frac{1}{66}$	$\frac{1}{6}+\frac{1}{66}$
13	$\frac{1}{8}+\frac{1}{52}+\frac{1}{104}$	$\frac{1}{7}+\frac{1}{91}$
15	$\frac{1}{10}+\frac{1}{30}$	$\frac{1}{8}+\frac{1}{120}$
67	$\frac{1}{40}+\frac{1}{335}+\frac{1}{536}$	$\frac{1}{34}+\frac{1}{2278}$
89	$\frac{1}{60}+\frac{1}{356}+\frac{1}{534}+\frac{1}{890}$	$\frac{1}{45}+\frac{1}{4005}$

Egyptian Fractions

Any positive rational a/n can be written as the sum of positive unit fractions

$$\frac{a}{n} = \frac{1}{m_1} + \frac{1}{m_2} + \cdots + \frac{1}{m_k}.$$

The above is an example of an Egyptian fraction decomposition of length k.

Here's an example of length 5:

$$\frac{867}{5309} = \frac{1}{7} + \frac{1}{49} + \frac{1}{23650} + \frac{1}{683592739} + \frac{1}{4205691294638106350}$$

Mathematical Questions from Egyptian Fractions

- How many terms do you need?
- How big are the denominators?
- What are the best choices for denominators?
- How long does the “greedy algorithm” take?
- How many representations of a/b as a sum of n Egyptian fractions are there?

Mathematicians working on these questions include Butler, Croot, Erdős, Graham, Konyagin, Yokota

Erdős–Straus conjecture

Conjecture

There exist positive integers m_1, m_2, m_3 such that

$$\frac{4}{n} = \frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3}.$$

For example:

$$\frac{4}{8675309} = \frac{1}{2168828} + \frac{1}{6271751022618} + \frac{1}{59002291334558621370338268}$$

- It has been verified for $n \leq 10^{14}$.
- The set of exceptions has density 0. (Vaughan 1970)

Averaging

Since we can't prove it, let's average!

Theorem (Elshotz–Tao (2013))

Let

$$f(n) = \#\left\{(m_1, m_2, m_3) \in \mathbb{N}^3 \,:\, \frac{4}{n} = \frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3}\right\},$$

then

$$x\log^2 x \ll \sum_{p\le x} f(p) \ll x\log^2 x\,\log\log x.$$

Length 2 and 3

$$A_k(n) = \#\left\{a \in \mathbb{N} : \frac{a}{n} = \frac{1}{m_1} + \frac{1}{m_2} + \cdots + \frac{1}{m_k}\right\}.$$

Theorem (Croot, Dobbs, Friedlander, Hetzel, Pappalardi (2000))

For any $\varepsilon > 0$,

$$x \log^4 x \ll \sum_{n \leq x} A_2(n) \ll x \log^4 x.$$

Theorem (Luca–Pappalardi (2019))

$$x \log^3 x \ll \sum_{p \leq x} A_3(p) \ll x \log^5 x.$$

$A_3(n)$

Theorem (Croot, Dobbs, Friedlander, Hetzel, Pappalardi (2000))

For any $\varepsilon > 0$,

$$A_3(n) \ll n^{\frac{1}{2}+\varepsilon}.$$

Theorem (Banderier, Gómez Ruiz, Luca, Pappalardi, Treviño)

Let $h(n) = C/\log\log n$, *where* $C = \frac{2\log(48)\log(\log(6983776800))}{\log(6983776800)} \approx 1.066$. *Then*

$$A_3(n) \leq 10n^{\frac{1}{2}+\frac{13}{4}h(n)}\log n.$$

Corollary

For $n \geq 10^{10^{23}}$,

$$A_3(n) \leq \frac{1}{100}n^{\frac{1}{2}+\frac{1}{15}}.$$

$F(n)$

$$F(n) = \#\left\{(a, m_1, m_2, m_3) \in \mathbb{N}^4 \;:\; \frac{a}{n} = \frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3}\right\}.$$

Theorem (Banderier, Gómez Ruiz, Luca, Pappalardi, Treviño)

Let $\varepsilon > 0$, then

$$F(n) \ll n^{\frac{5}{6}+\varepsilon}.$$

This implies that for large enough n, $F(n) < n$. This suggests the question, what is the largest n such that $F(n) \geq n$.
The first values for which $F(n) < n$ are:
$F(8821) = 8590, F(11161) = 10270, F(11941) = 10120.$

Explicit $F(n)$

Theorem (Banderier, Gómez Ruiz, Luca, Pappalardi, Treviño)

For $n \geq 10^{10^{23}}$.

$$F(n) \leq \frac{1}{10} n$$

Number Theory in the Americas

Parametrization Lemma

Lemma (Luca, Pappalardi)

Consider an Egyptian fraction decomposition of the irreducible fraction a/n:

$$\frac{a}{n} = \frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} \qquad \textit{with } \gcd(a,n) = 1 \tag{1}$$

Then there exist integers $D_1, D_2, D_3, v_1, v_2, v_3$ with

(i) $lcm(D_1, D_2, D_3) \mid n$ *and* $\gcd(D_1, D_2, D_3) = 1$;

(ii) $av_1v_2v_3 \mid D_1v_1 + D_2v_2 + D_3v_3$ *and* $\gcd(v_i, D_jv_j) = 1$ *when* $i \neq j$,

and the denominators of the Egyptian fractions are given by

$$m_i = \frac{n(D_1v_1 + D_2v_2 + D_3v_3)}{aD_iv_i}. \tag{2}$$

Conversely, if conditions (i)–(ii) are fulfilled, then the m_i's defined via (2) are integers, and denominators of k unit fractions summing to a/n.

Thank you!