The Least Inert Prime in a Real Quadratic Field

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An upperbound on the least inert prime in a real quadratic field

An integer D is a fundamental discriminant if and only if either D is squarefree, $D \neq 1$, and $D \equiv \pmod{4}$ or $D = 4L$ with L squarefree and $L \equiv 2, 3 \pmod{4}$.

Theorem (Granville, Mollin and Williams, 2000)

For any positive fundamental discriminant $D > 3705$, there is always at least one prime $p \leq \sqrt{D}/2$ such that the Kronecker symbol $(D/p) = -1$.

Improved upperbound

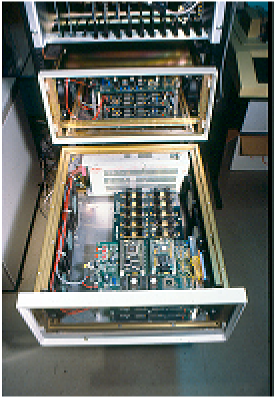
Theorem (ET, 2010)

For any positive fundamental discriminant $D > 1596$, there is always at least one prime $p \leq D^{0.45}$ such that the Kronecker symbol $(D/p) = -1$.

Elements of the Proof

- Use a computer to check the “small” cases. Granville, Mollin and Williams used the Manitoba Scalable Sieving Unit.
- Use analytic techniques to prove it for the “infinite case”, i.e. the very large D. The tool used by Granville et al. was the Pólya–Vinogradov inequality. I used a “smoothed” version of it.
- Use Pólya–Vinogradov plus a bit of clever computing to fill in the gap.

Manitoba Scalable Sieving Unit



Pólya–Vinogradov

Let χ be a Dirichlet character to the modulus $q > 1$. Let

$$S(\chi) = \max_{M,N} \left| \sum_{n=M+1}^{M+N} \chi(n) \right|$$

The Pólya–Vinogradov inequality (1918) states that there exists an absolute universal constant c such that for any Dirichlet character $S(\chi) \le c\sqrt{q}\log q$.

Under GRH, Montgomery and Vaughan showed that $S(\chi) \ll \sqrt{q}\log\log q$.

Paley showed in 1932 that there are infinitely many quadratic characters such that $S(\chi) \gg \sqrt{q}\log\log q$.

Further results regarding Pólya–Vinogradov

Granville and Soundararajan showed that one can save a small power of $\log q$ in the Pólya–Vinogradov inequality. Goldmakher improved it to

Theorem (Goldmakher, 2007)

For each fixed odd number $g > 1$, for $\chi \pmod q$ of order g,

$$S(\chi) \ll_g \sqrt{q}(\log q)^{\Delta_g + o(1)}, \quad \Delta_g = \frac{g}{\pi}\sin\frac{\pi}{g}, \quad q \to \infty.$$

Moreover, under GRH

$$S(\chi) \ll_g \sqrt{q}(\log\log q)^{\Delta_g + o(1)}.$$

Furthermore, there exists an infinite family of characters $\chi \pmod q$ of order g satisfying

$$S(\chi) \gg_{\epsilon,g} \sqrt{q}(\log\log q)^{\Delta_g - \epsilon}.$$

Asymptotic results on least inert primes in a real quadratic field

- Using the Pólya–Vinogradov, it easily follows that there exists a $p \ll \sqrt{D} \log D$ such that $\left(\frac{D}{p}\right) = -1$.
- By using a little sieving, we can improve this result: For every $\epsilon > 0$, there exists a prime $p \ll_\epsilon D^{\frac{1}{2\sqrt{e}}+\epsilon}$ such that $\left(\frac{D}{p}\right) = -1$.
- Using the Burgess inequality and a little sieving, we get the best unconditional result we have now: For every $\epsilon > 0$, there exists a prime $p \ll_\epsilon D^{\frac{1}{4\sqrt{e}}+\epsilon}$ such that $\left(\frac{D}{p}\right) = -1$.

Burgess

Theorem (Burgess, 1962)

Let χ be a primitive character mod q with $q > 1$, r an integer and $\epsilon > 0$ a real number. Then

$$S(\chi) \ll_{\epsilon,r} N^{1-\frac{1}{r}} q^{\frac{r+1}{4r^2}+\epsilon}$$

for $r = 2, 3$ and for any $r \geq 1$ if q is cubefree, the implied constant depending only on ϵ and r.

Explicit Pólya–Vinogradov

Theorem (Hildebrand, 1988)

For χ a primitive character to the modulus $q > 1$, we have

$$|S(\chi)| \leq \begin{cases} \left(\frac{2}{3\pi^2} + o(1)\right)\sqrt{q}\log q & , \quad \chi \text{ even}, \\ \left(\frac{1}{3\pi} + o(1)\right)\sqrt{q}\log q & , \quad \chi \text{ odd}. \end{cases}$$

Theorem (Pomerance, 2009)

For χ a primitive character to the modulus $q > 1$, we have

$$|S(\chi)| \leq \begin{cases} \frac{2}{\pi^2}\sqrt{q}\log q + \frac{4}{\pi^2}\sqrt{q}\log\log q + \frac{3}{2}\sqrt{q} & , \quad \chi \text{ even}, \\ \frac{1}{2\pi}\sqrt{q}\log q + \frac{1}{\pi}\sqrt{q}\log\log q + \sqrt{q} & , \quad \chi \text{ odd}. \end{cases}$$

Explicit Burgess

Theorem (Iwaniec-Kowalski-Friedlander)

Let χ be a Dirichlet character mod p (a prime). Then for $r \geq 2$

$$|S_\chi(N)| \leq 30 \cdot N^{1-\frac{1}{r}} p^{\frac{r+1}{4r^2}} (\log p)^{\frac{1}{r}}.$$

Theorem (ET, 2009)

Let χ be a Dirichlet character mod p (a prime). Then for $r \geq 2$ and $p \geq 10^7$.

$$|S_\chi(N)| \leq 3 \cdot N^{1-\frac{1}{r}} p^{\frac{r+1}{4r^2}} (\log p)^{\frac{1}{r}}.$$

Note, the constant gets better for larger r, for example for $r = 3, 4, 5, 6$ the constant is $2.376, 2.085, 1.909, 1.792$ respectively.

Quadratic Case for Burgess

Theorem (Booker, 2006)

Let $p > 10^{20}$ *be a prime number* $\equiv 1 \pmod 4$, $r \in \{2, \dots, 15\}$ *and* $0 < M, N \leq 2\sqrt{p}$. *Let* χ *be a quadratic character* $\pmod p$. *Then*

$$\left| \sum_{M \leq n < M+N} \chi(n) \right| \leq \alpha(r) p^{\frac{r+1}{4r^2}} (\log p + \beta(r))^{\frac{1}{2r}} N^{1-\frac{1}{r}}$$

where $\alpha(r), \beta(r)$ *are given by*

r	$\alpha(r)$	$\beta(r)$	r	$\alpha(r)$	$\beta(r)$
2	1.8221	8.9077	9	1.4548	0.0085
3	1.8000	5.3948	10	1.4231	-0.4106
4	1.7263	3.6658	11	1.3958	-0.7848
5	1.6526	2.5405	12	1.3721	-1.1232
6	1.5892	1.7059	13	1.3512	-1.4323
7	1.5363	1.0405	14	1.3328	-1.7169
8	1.4921	0.4856	15	1.3164	-1.9808

Some Applications of the Explicit Estimates

- Norton showed that for every prime p, it's least quadratic non-residue is $\leq 4.7p^{1/4} \log p$.
- For computing class numbers of large discriminants. Booker, computed the class number of a 32-digit discriminant.
- To prove a conjecture of Brizolis (Levin, Pomerance) that for every prime $p > 3$ there is a primitive root g and an integer $x \in [1, p-1]$ with $\log_g x = x$, that is, $g^x \equiv x \pmod{p}$.

Smoothed Pólya–Vinogradov

Let M, N be real numbers with $0 < N \leq q$, then define $S^*(\chi)$ as follows:

$$S^*(\chi) = \max_{M,N} \left| \sum_{M \leq n \leq 2N} \chi(n) \left(1 - \left| \frac{a - M}{N} - 1 \right| \right) \right|.$$

Theorem (Levin, Pomerance, Soundararajan, 2009)

Let χ be a primitive character to the modulus $q > 1$, and let M, N be real numbers with $0 < N \leq q$, then

$$S^*(\chi) \leq \sqrt{q} - \frac{N}{\sqrt{q}}.$$

Lowerbound for the smoothed Pólya–Vinogradov

Theorem (ET, 2010)

Let χ be a primitive character to the modulus $q > 1$, and let M, N be real numbers with $0 < N \leq q$, then

$$S^*(\chi) \geq \frac{2}{\pi^2}\sqrt{q}.$$

Therefore, the order of magnitude of $S^*(\chi)$ is $\sqrt{q}$.

A little background on the smoothed Pólya–Vinogradov

L.K. Hua had proved an equivalent statement for prime modulus and used it to give an upperbound for the least primitive root.

Theorem (Hua, 1942)

Let $p > 2$, $1 \leq A < (p-1)/2$. Then, for each non-principal character, $\bmod p$, *we have*

$$\frac{1}{A+1}\left|\sum_{a=0}^{A}\sum_{n=A+1-a}^{A+1+a}\chi(n)\right| \leq \sqrt{p} - \frac{A+1}{\sqrt{p}}.$$

Manitoba Scalable Sieving Unit

Recall that we are dealing with D a fundamental discriminant, i.e. either $D = L$ or $D = 4L$ where L is squarefree. We only need to consider the cases $D \equiv 1 \pmod 8$ and $D \equiv 2, 3 \pmod 4$ because $D/2) = -1$ for $D \equiv 5 \pmod 8$.

Running the Manitoba Scalable Sieving Unit (MSSU) for about 5 months yielded, among other things, the following information: If

1. $L \equiv 1 \pmod 8$ with $(L/q) = 0$ or 1 for all odd $q \leq 257$,
2. $L \equiv 2 \pmod 4$ with $(L/q) = 0$ or 1 for all odd $q \leq 283$ or
3. $L \equiv 3 \pmod 4$ with $(L/q) = 0$ or 1 for all odd $q \leq 277$

then $L > 2.6 \times 10^{17}$.

Counterexamples

The MSSU then allows us to know that we need only check up to $4(283)^2 = 320356$ for counterexamples below 2.6×10^{17} (or $4 \times 2.6 \times 10^{17}$ in the case of D even), for least inert primes $> \sqrt{D}/2$. The set of counterexamples is

$$S = \{5, 8, 12, 13, 17, 24, 28, 33, 40, 57, 60, 73, 76, 88, 97, 105, 120, 124, 129, 136, 145, 156, 184, 204, 249, 280, 316, 345, 364, 385, 424, 456, 520, 561, 609, 616, 924, 940, 984, 1065, 1596, 2044, 3705\}.$$

Similarly for the counterexamples to least inert prime $> D^{0.45}$, we need only check up to $283^{1/.45} = 280811$. The set of counterexamples is

$$S' = \{8, 12, 24, 28, 33, 40, 60, 105, 120, 156, 184, 204, 280, 364, 456, 520, 1596\}.$$

Tighter smoothed PV

Theorem (ET, 2010)

Let χ be a primitive character to the modulus $q > 1$, let M, N be real numbers with $0 < N \leq q$. Then

$$\left| \sum_{M \leq n \leq M+2N} \chi(n) \left(1 - \left|\frac{n-M}{N} - 1\right|\right) \right| \leq \frac{\phi(q)}{q}\sqrt{q} + 2^{\omega(q)-1}\frac{N}{\sqrt{q}}.$$

Applying smoothed PV to the infinite case

Let $\chi(p) = \left(\frac{D}{p}\right)$. Since D is a fundamental discriminant, χ is a primitive character of modulus D. Consider

$$S_\chi(N) = \sum_{n \leq 2N} \chi(n) \left(1 - \left|\frac{n}{N} - 1\right|\right).$$

By smoothed PV, we have

$$|S_\chi(N)| \leq \frac{\phi(D)}{D}\sqrt{D} + 2^{\omega(D)-1}\frac{N}{\sqrt{D}}.$$

Now,

$$S_\chi(N) = \sum_{\substack{n \le 2N \\ (n,D)=1}} \left(1 - \left|\frac{n}{N} - 1\right|\right) - 2 \sum_{\substack{B < p \le 2N \\ \chi(p)=-1}} \sum_{\substack{n \le \frac{2N}{p} \\ (n,D)=1}} \left(1 - \left|\frac{np}{N} - 1\right|\right).$$

Therefore,

$$\frac{\phi(D)}{D}\sqrt{D} + 2^{\omega(D)-1}\frac{N}{\sqrt{D}} \ge |S_\chi(N)| \ge \frac{\phi(D)}{D}N - 2^{\omega(D)-2} - 2 \sum_{\substack{n \le \frac{2N}{B} \\ (n,D)=1}} \sum_{B < p \le \frac{2N}{n}} \left(1 - \left|\frac{np}{N} - 1\right|\right).$$

Now, letting $N = c\sqrt{D}$ for some constant c we get

$$0 \ge c - 1 - 2^{\omega(D)}\left(\frac{c}{2} + \frac{1}{4}\right)\frac{D}{\phi(D)\sqrt{D}} - \frac{2}{\sqrt{D}}\frac{D}{\phi(D)} \sum_{\substack{n \le \frac{2N}{B} \\ (n,D)=1}} \sum_{B < p \le \frac{2N}{n}} \left(1 - \left|\frac{np}{N} - 1\right|\right)$$

Eventually we have,

$$0 \geq c - 1 - 2^{\omega(D)} \left(\frac{c}{2} + \frac{1}{4} \right) \frac{D}{\phi(D)\sqrt{D}} - \frac{2c}{\log B} e^{\gamma} \left(1 + \frac{1}{\log^2\left(\frac{2N}{B}\right)} \right) \log\left(\frac{2N}{B}\right) \prod_{\substack{p > \frac{2N}{B} \\ p \mid D}} \frac{p}{p-1}.$$

For $D \geq 10^{24}$ this is a contradiction.

Hybrid Case

We have as in the previous case

$$0 \geq c - 1 - 2^{\omega(D)} \left(\frac{c}{2} + \frac{1}{4} \right) \frac{D}{\phi(D)\sqrt{D}} - \frac{2}{\sqrt{D}} \frac{D}{\phi(D)} \sum_{\substack{n \leq \frac{2N}{B} \\ (n,D)=1}} \sum_{B < p \leq \frac{2N}{n}} \left(1 - \left| \frac{np}{N} - 1 \right| \right)$$

In this case, since we don't have to worry about the infinite case, we can have a messier version of

$$\sum_{B < p \leq \frac{2N}{n}} \left(1 - \left| \frac{np}{N} - 1 \right| \right).$$

The idea is to consider 2^{13} cases, one for each possible value of (D, M) where $M = \prod_{p \leq 41} p$.

- We consider the odd values and the even values separately. For odd values, the strategy of checking all the cases proves the theorem for $21853026051351495 = 2.2\ldots \times 10^{16}$.
- For even values we get the theorem for $1707159924755154870 = 1.71\ldots \times 10^{18}$.
- Here we need a little extra work, we find that there are 12 outstanding cases and we deal with them one at a time.
- QED.

Future Work

- Bringing the upperbound further down.
- Generalizing to D's not necessarily fundamental discriminants.
- Generalizing to other characters, not just the Kronecker symbol.
- Extending the explicit Burgess results to other modulus, not just prime modulus.

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