Many proofs that $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$

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Theorem 1. Let n be a positive integer. Then

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

1. **Induction**:
   Since $1(2)/2 = 1$, then it is true for $n = 1$. Now suppose it's true for n. Then

   $$1+2+\cdots+n+(n+1) = (1+\cdots+n)+(n+1) = \frac{n(n+1)}{2} + (n+1) = \frac{n+1}{2}(n+2).$$

2. **Gauss trick**:
   Let $S = 1 + 2 + \cdots + n$. Then

   $$\begin{aligned} S &= 1 + 2 + \cdots + (n-1) + n \\ S &= n + (n-1) + \cdots + 2 + 1, \end{aligned}$$

   so

   $$2S = (n+1) + (n+1) + \cdots + (n+1) = n(n+1).$$

   The result follows.

   Remark 2. This technique can be generalized to add numbers that are in an arithmetic progression. Indeed, suppose we have the arithmetic progression $a, a+b, a+2b, \ldots, a+(n-1)b$ and we want to add them. Then

   $$S = a + (a+b) + \cdots + (a + (n-1)b).$$

   Doing the same idea, we get $2S = n(a + a + (n-1)b)$. In other words, the sum is the number of terms (n) times the sum of the first and last term $(a+(a+(n-1)b))$ divided by 2.

3. **Gauss trick variant**:
   Suppose n is even, i.e., $n = 2k$. Then we can group the sum as follows:

   $$1+2+\cdots+n = (1+2k)+(2+(2k-1))+\cdots+(k+(k+1)) = k(2k+1) = \frac{n(n+1)}{2}.$$

   Suppose n is odd, i.e., $n = 2k+1$. Then we can group the sum as

   $$(1+(2k))+(2+(2k-1))+\cdots+(k+(k+1))+(2k+1) = (2k+1)(k+1) = \frac{n(n+1)}{2}.$$

4. **Probabilistic Proof**:
Let X be the sum of two n-sided dice. For $k = 2, 3, \dots, n+1$, the probability that $X = k$ is $\frac{(k-1)}{n^2}$ because there are $k-1$ ways of adding two positive integers to k, namely $1+(k-1), 2+(k-2), \cdots, (k-1)+1$. For $k = n+i$ with $i = 2, 3, \cdots, n$, the probability that $X = k$ is $\frac{n-i+1}{n^2}$ because there are $n-i+1$ ways of adding up to k with two numbers from $\{1, 2, \cdots, n\}$, namely $i+n, (i+1)+(n-1), \cdots, n+i$. Since $2 \leq X \leq 2n$, then

$$\begin{aligned}
1 &= \sum_{k=2}^{2n} \mathbb{P}[X = k] = \sum_{k=2}^{n+1} \frac{k-1}{n^2} + \sum_{i=2}^{n} \frac{n-i+1}{n^2} \\
1 &= \left(\frac{1}{n^2} + \frac{2}{n^2} + \cdots + \frac{n}{n^2}\right) + \left(\frac{n-1}{n^2} + \frac{n-2}{n^2} + \cdots + \frac{1}{n^2}\right) \\
n^2 &= (1 + 2 + \cdots + n) + (1 + 2 + \cdots + (n-1)) \\
n^2 &= 2(1 + 2 + \cdots + n) - n.
\end{aligned}$$

Hence,

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

Remark 3. This proof appeared in [19].

5. **Writing as a Double Sum**:

$$\sum_{i=1}^{n} i = \sum_{i=1}^{n} \sum_{k=0}^{i-1} 1 = \sum_{0 \leq k < i \leq n} 1.$$

The last sum is a combinatorial problem. We want to count the number of ways of picking integers k, i satisfying $0 \leq k < i \leq n$. But that is simply $\binom{n+1}{2}$ since it boils down to choosing two numbers from $\{0, 1, \dots, n\}$. The proof follows from the fact that $\binom{n+1}{2} = \frac{n(n+1)}{2}$.

Remark 4. This technique has been generalized in [18] to find formulas for $S_k(n) = 1^k + 2^k + \cdots + n^k$. For example, for the sum of squares, we have

$$\sum_{c \leq n} c^2 = \sum_{c \leq n} \sum_{a \leq c} \sum_{b \leq c} 1 = \sum_{1 \leq a, b \leq c \leq n} 1.$$

That is, we want to find the number of triples (a, b, c) satisfying that $1 \leq a \leq c$ and $1 \leq b \leq c$. We can split this into the following cases: $(a < b < c), (a < b = c), (a = b < c), (a = b = c)$. By symmetry of a and b we can just multiply the cases of the form $(a < b)$ by 2. Then, we have

$$\begin{aligned}
\sum_{c=1}^{n} c^2 &= 2\binom{n}{3} + 2\binom{n}{2} + \binom{n}{2} + \binom{n}{1} = 2\binom{n}{3} + 3\binom{n}{2} + \binom{n}{1} \\
&= \frac{n}{6}\left(2(n-1)(n-2) + 9(n-1) + 6\right) = \frac{n(n+1)(2n+1)}{6}.
\end{aligned}$$

6. **Generating Functions**:
Let $a_n = 1 + 2 + \cdots + n$ and let

$$A(n) = \sum_{n=0}^{\infty} a_n x^n.$$

Since $a_n = a_{n-1} + n$, then

$$A(n) = 0 + \sum_{n=1}^{\infty}(a_{n-1} + n)x^n = \sum_{n=1}^{\infty} a_{n-1}x^n + \sum_{n=1}^{\infty} nx^n = x\sum_{n=0}^{\infty} a_n x^n + \sum_{n=1}^{\infty} nx^n. \tag{1}$$

We have $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$, therefore

$$\sum_{n=1}^{\infty} nx^{n-1} = \frac{1}{(1-x)^2}, \tag{2}$$

which implies that

$$\sum_{n=1}^{\infty} nx^{n-1} = \frac{1}{(1-x)^2}.$$

From this it follows that

$$\sum_{n=0}^{\infty} nx^n = x\sum_{n=1}^{\infty} nx^{n-1} = \frac{x}{(1-x)^2}.$$

Therefore, using this in (1) yields

$$A(n) = xA(n) + \frac{x}{(1-x)^2}.$$

Therefore

$$A(n) = \frac{x}{(1-x)^3}.$$

But, taking the derivative of (2) reveals that

$$\sum_{n=2}^{\infty} n(n-1)x^{n-2} = \frac{2}{(1-x)^3}.$$

Therefore

$$\frac{x}{(1-x)^3} = \sum_{n=2}^{\infty} \frac{n(n-1)}{2}x^{n-1} = \sum_{n=1}^{\infty} \frac{n(n+1)}{2}x^n.$$

Therefore $a_n = \frac{n(n+1)}{2}$.

7. **Counting Two Different Ways**:
Consider the set $\{1, 2, \ldots, n+1\}$. We will count how many pairs of numbers come from that set in two different ways. On the one hand, the answer is $\binom{n+1}{2}$. On the other hand, suppose the pair is (a, b) with $a < b$. Since b is the largest of the two, $2 \leq b \leq n+1$. Fix b, then there are $b-1$ ways of picking a. Since $2 \leq b \leq n+1$ we have

$$\binom{n+1}{2} = \sum_{b=2}^{n+1}(b-1) = 1 + 2 + \cdots + n.$$

Remark 5. The proof above can be generalized as follows. Suppose you want to choose $r+1$ things from the set $\{1, 2, \ldots, n+1\}$. On the one hand, there are $\binom{n+1}{r+1}$ ways of doing that. On the other, if we fix the largest element b of a $(r+1)-\textit{subset}$, that element satisfies $r+1 \le b \le n+1$ and there are $\binom{b-1}{r}$ ways of choosing the rest. Therefore, we get what is known as the Christmas Stocking Theorem (or Hockey Stick Identity)

$$\binom{r}{r} + \binom{r+1}{r} + \cdots + \binom{n}{r} = \binom{n+1}{r+1}.$$

8. **Interpolation**:
Suppose the answer is a quadratic $f(n) = an^2+bn+c$. Then the quadratic should satisfy $f(0) = 0$, so $c = 0$. It should also satisfy $f(1) = 1$, $f(2) = 3$, so $a + b = 1, 4a + 2b = 3$. From this we deduce that $a = b = 1/2$. Therefore $f(n) = \frac{n^2+n}{2} = \frac{n(n+1)}{2}$. One can then use induction to conclude.

9. **Difference Operator**:
Let $f(n) = 1 + 2 + \cdots + n$. We can spot the recursion $f(n) = f(n-1) + n$, therefore $R(n) = f(n) - f(n-1) = n$ is a linear function. But the degree of $R(n)$ is exactly one less than the degree of $f(n)$. Therefore $f(n)$ is a polynomial of degree 2. We can deduce that $f(n) = \frac{n^2}{2} + \frac{n}{2}$ as done in Proof 8.

Remark 6. This argument can be used to show that $S_k(n) = 1^k + 2^k + \cdots + n^k$ must be a polynomial of degree $k+1$.

10. **Pascal Proof**:
Note that $(n+1)^2 - n^2 = 2n + 1$. Therefore

$$\begin{aligned}
(n+1)^2 - n^2 &= 2n + 1 \\
n^2 - (n-1)^2 &= 2(n-1) + 1 \\
(n-1)^2 - (n-2)^2 &= 2(n-2) + 1 \\
&\vdots \\
2^2 - 1^2 &= 2(1) + 1.
\end{aligned}$$

By adding up all the columns, we get that the left side telescopes, so

$$(n+1)^2 - 1 = 2(n + (n-1) + \cdots + 1) + n,$$

so

$$1 + 2 + \cdots + n = \frac{(n+1)^2 - 1 - n}{2} = \frac{n^2 + n}{2} = \frac{n(n+1)}{2}.$$

Remark 7. This technique can be generalized to find a formula for $S_k(n) = 1^k + 2^k + \cdots + n^k$. By using that

$$(n+1)^{k+1} - 1 = \sum_{\ell=0}^{k} \binom{k+1}{\ell} S_\ell(n),$$

and telescoping, we get Pascal's identity [14]:

$$S_k(n) = \frac{(n+1)^{k+1} - 1 - \sum_{\ell=0}^{k-1} \binom{k+1}{\ell} S_\ell(n)}{k+1}.$$

11. **Geometric Series + L'Hospital**:
Consider the identity:
$$\sum_{k=0}^{n} x^k = \frac{1 - x^{n+1}}{1 - x}.$$
Take the derivative on both sides to get
$$\sum_{k=1}^{n} kx^{k-1} = \frac{-(1-x)(n+1)x^n + (1 - x^{n+1})}{(1-x)^2} = \frac{nx^{n+1} - (n+1)x^n + 1}{(1-x)^2}.$$
Now let's take the limit as $x \to 1$. On the left side we get $1 + 2 + \cdots + n$. The right side we evaluate using L'Hospital to get
$$\begin{aligned}\lim_{x\to 1} \frac{nx^{n+1} - (n+1)x^n + 1}{(1-x)^2} &= \lim_{x\to 1} \frac{n(n+1)x^n - (n+1)nx^{n-1}}{-2(1-x)} \\ &= \lim_{x\to 1} \frac{n^2(n+1)x^{n-1} - (n+1)n(n-1)x^{n-2}}{2} \\ &= \frac{n(n+1)}{2}(n - (n-1)) = \frac{n(n+1)}{2}.\end{aligned}$$
Remark 8. This proof was suggested to the second author by Steven J. Miller. The techniques here can be generalized to get higher powers.

12. **Area Proof**:
Imagine each number k represented by a row of k unit squares. Then the sum $1 + 2 + \cdots + n$ forms a shape similar to a triangle as in Figure 1. If we copy it, rotate it and glue the two pieces together, we get an $n \times (n+1)$ rectangle as in Figure 2. Therefore, the overall area is $n(n+1)/2$.

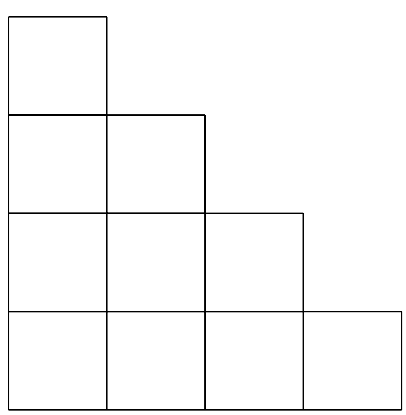


Figure 1: The sum as an area of a shape.

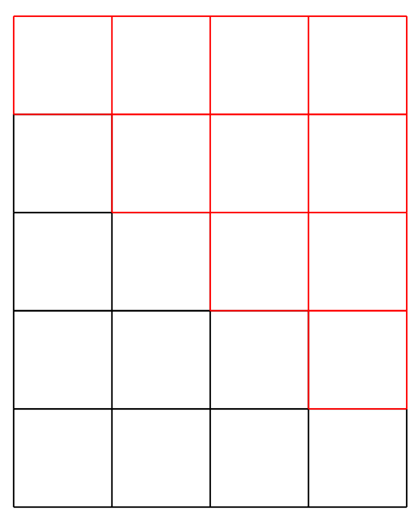


Figure 2: The two pieces forming a rectangle.

13. **Another Area Proof**:
As in Proof 14, we want to evaluate the area of Figure 1. In Figure 4, we can see that there's a triangle of area $\frac{n^2}{2}$ and n triangles of area $1/2$. Therefore, the area is $\frac{n^2+n}{2}$.

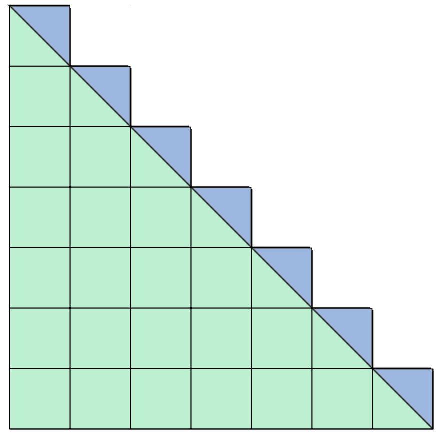


Figure 3: Proof looking at areas.

14. **Another Area Proof**:
    This visual was suggested by Bob Palais. Again let each number k be represented by a row of k unit squares so that the sum $1 + 2 + \cdots + n$ forms the triangular array as in Figure 4. If we use two copies of this array with a diagonal line of squares inserted, we get an $(n+1) \times (n+1)$ square as in Figure 5. This diagram pictorially shows that

$$(n+1)^2 = 2\sum_{i=1}^{n} i + (n+1)$$

$$2\sum_{i=1}^{n} i = (n+1)^2 - (n+1) = n^2 + n.$$

    This image and proof is related to the Pascal proof previously given.

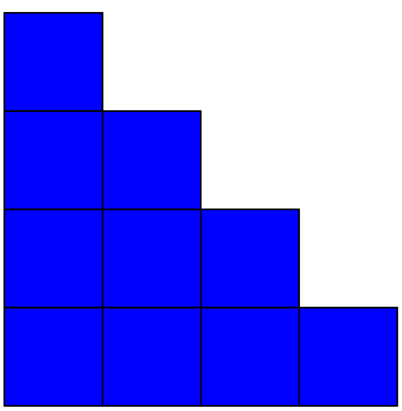


Figure 4: The sum as an area of a shape.

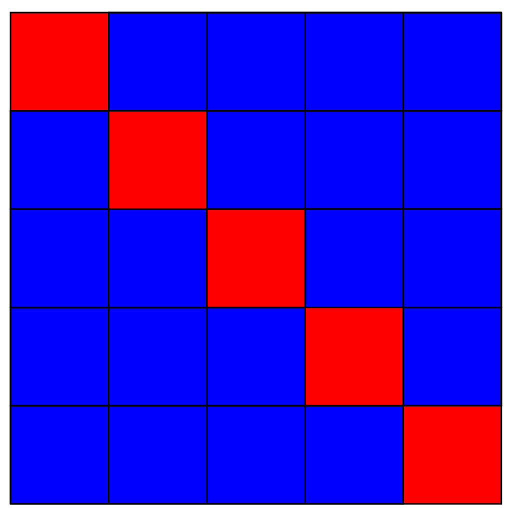


Figure 5: Two arrays and a line form a square.

15. **Trapezoid Area**:
    Figure 6 contains a proof without words depending on evaluating the area of a trapezoid in two ways. The original figure is from Gaspar [6].

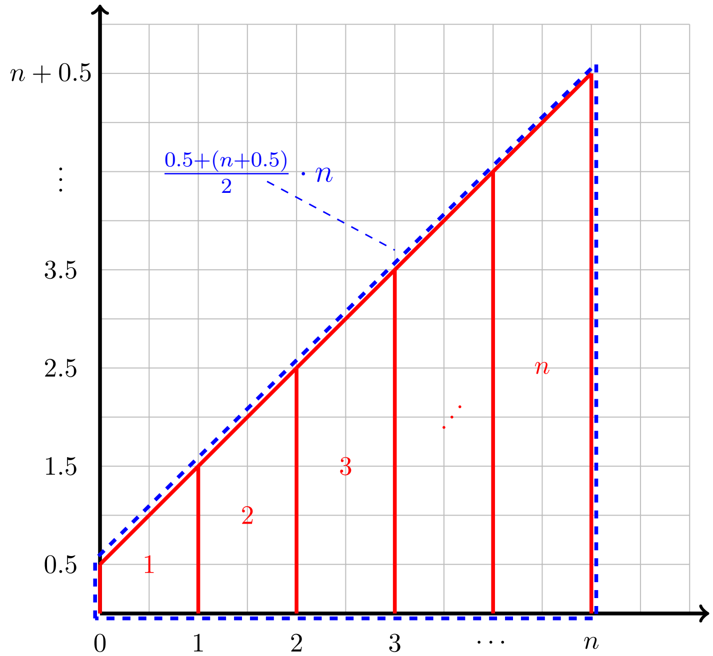


Figure 6: The area of the trapezoid.

16. **Integral Version of Trapezoid Area**:
As detailed in [6], Proof 15 can be translated into the following proof with integrals: We have, for $k \geq 1$,

$$\int_{k-1}^{k} \left(\frac{1}{2}+x\right) \, \mathrm{d}x = \left(\frac{x+x^2}{2}\right)\Big|_{k-1}^{k} = k.$$

Therefore

$$\int_{0}^{n} \left(\frac{1}{2}+x\right) \, \mathrm{d}x = \sum_{k=1}^{n} \int_{k-1}^{k} \left(\frac{1}{2}+x\right) \, \mathrm{d}x = \sum_{k=1}^{n} k.$$

But

$$\int_{0}^{n} \left(\frac{1}{2}+x\right) \, \mathrm{d}x = \left(\frac{x+x^2}{2}\right)\Big|_{0}^{n} = \frac{n^2+n}{2}.$$

17. **Handshakes in a room**:
Consider $n+1$ persons in a room and every pair of people greet each other with a handshake. Let's answer the question of how many handshakes were given in two ways. On the one hand, there should be $\binom{n+1}{2}$ ways, since there's a handshake for every pair. On the other hand, if we label the people as $P_1, P_2, \ldots, P_{n+1}$, and we count in a orderly way, we see that P_1 gave n handshakes, then P_2 gave $n-1$ other handshakes (the handshake with P_1 is already counted), then P_3 gave $n-2$ other handshakes, and so on. Therefore $\binom{n+1}{2} = n + (n-1) + \cdots + 1$.

We note that this proof is equivalent to counting the number of edges in the complete graph on $n+1$ vertices, K_{n+1}. A diagram following [3] describing the idea for the previous argument in this setting is in Figure 7 when $n=5$.

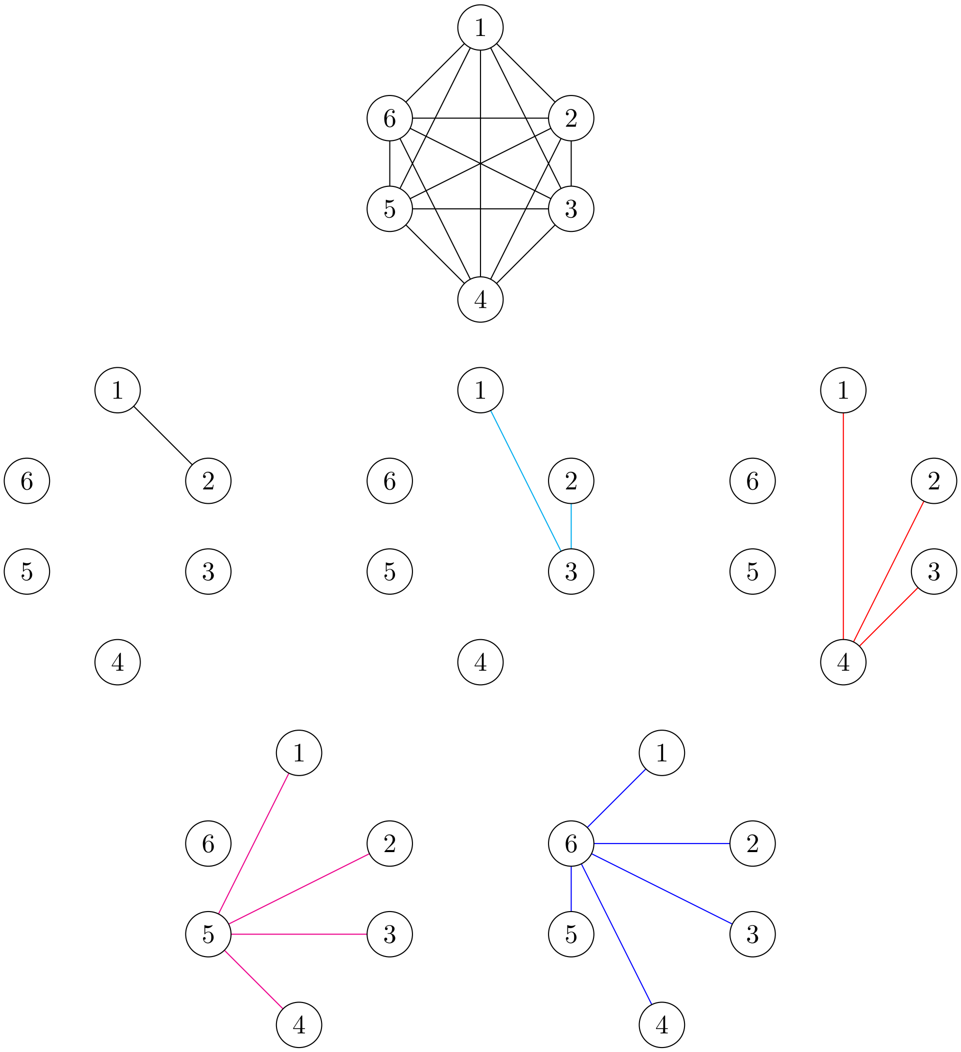


Figure 7: Counting edges in the complete graph on n vertices.

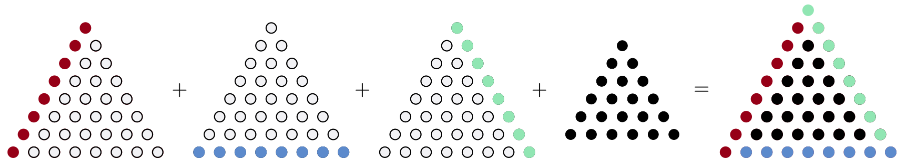
18. **Stars and Bars proof**:
We want to find the number of solutions to the equation $x_1 + x_2 + x_3 = n - 1$ with $x_1, x_2, x_3 \geq 0$. We can imagine a word consisting of $n - 1$ stars and 2 bars. Then x_1 is the number of stars before the first bar, x_2 is the number of stars between the first bar and the second, and x_3 is the number of bars after the second bar. For example, when $n = 5$ we would have $\star|\star\star\star|\star$ representing $x_1 = 1$, $x_2 = 3$, $x_3 = 1$, and $\star||\star\star\star\star$ representing $x_1 = 1$, $x_2 = 0$, $x_3 = 4$. We can see that there's a bijection between the number of words with $n - 1$ stars and 2 bars with the number of non-negative integer solutions to $x_1 + x_2 + x_3 = n - 1$. Therefore, the number of solutions is $\binom{n+1}{2}$. However, we could also count it a different way. Fix x_1. Now $x_2 + x_3 = n - 1 - x_1$. Once x_2 is chosen between 0 and $n-1-x_1$, then x_3 is fixed. Therefore, for each x_1, there are $n - x_1$ solutions to the equation $x_1 + x_2 + x_3 = n - 1$. But $0 \leq x_1 \leq n - 1$, so we have that the number of solutions is $(n - 0) + (n - 1) + \cdots + (n - (n - 1)) = n + (n - 1) + \cdots + 1$.

19. **Via recurrence relations**:

The first thing to note is that the sequence $t_n := 1 + 2 + 3 + \cdots + n$ satisfies a linear homogeneous recurrence.

Theorem 9. The triangular numbers, $t_n := 1 + 2 + 3 + \cdots + n$, satisfy the recursion $t_n = 3t_{n-1} - 3t_{n-2} + t_{n-3}$.

Proof.



$\square$

The visual proof above comes from Edgar [5].

Then, this means that the characteristic equation for this sequence is $x^3 - 3x^2 + 3x - 1 = (x-1)^3$. Since this has the triple repeated root $x = 1$, we know that

$$t_n = a(1)^n + bn(1)^n + cn^2(1)^n.$$

The proof follows as in Proof 8, i.e., using the fact that $t_0 = 0$, $t_1 = 1$ and $t_2 = 3$, we see that $a = 0$, $b = \frac{1}{2}$ and $c = \frac{1}{2}$.

20. **Via the sum of cubes**

We will prove it using that

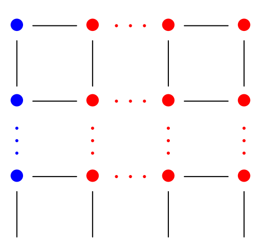
$$(1 + 2 + 3 + \cdots + n)^2 = 1^3 + 2^3 + 3^3 + \cdots n^3. \tag{3}$$

There is a famous visual proof of this due to J. Barry Love [11]. Love's picture (recreated by us) can be found on Figure 8.

Now we will count the number of rectangles inside the $(n+1) \times (n+1)$ grid in two ways. Let R_n be the number of rectangles in the $n \times n$ grid; we can check that $R_1 = 0$, $R_2 = 1$, $R_3 = 9$, and $R_4 = 36$.

Suppose $n > 1$. We utilize the fact that there will be an $n \times n$ grid sitting in the upper right of the $(n+1) \times (n+1)$ grid. We color the dots in the left column blue and the dots in the bottom row green, except for the bottom left corner, which we leave black. Once again, we say that a rectangle *uses a dot* if that dot appears as a corner of the rectangle.

In the model for the $(n+1) \times (n+1)$ grid, we note that there is a red $n \times n$ grid giving us n^2 red dots, there are n blue dots and n green dots.



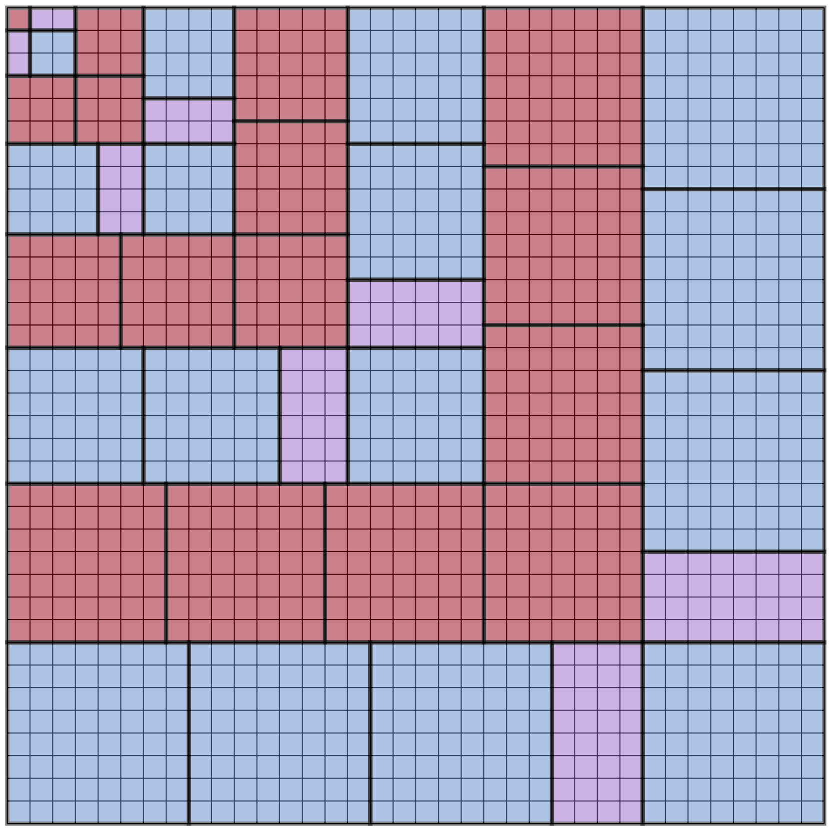
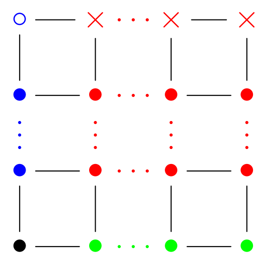


Figure 8: Proof that $\sum_{k=1}^{n} k^3 = \left(\sum_{k=1}^{n} k\right)^2$.

Now, we know that there are R_n rectangles completely contained in the grid of red dots. Again, the remaining rectangles fall into three distinct, non-overlapping classes: the rectangles that use two blue and two red dots, the rectangles that use two green and two red dots, and the rectangles that use the black dot. Also, we note that the following important observation holds.

* The number of rectangles using two blue dots and two red dots is the same as the number of rectangles using two green dots and two red dots.

Thus, we must first count the number of rectangles using two blue dots and two red dots. We see that for each blue dot, we must determine the red dots that we can use as the opposite corner of a possible rectangle. Fix one of the blue dots. Then we see that there are $n \cdot (n-1)$ red dots that we can use as the opposite corner of a rectangle. For instance, if we fix the top blue dot, $\circ$, we cannot use any of the red dots in the top row as the opposite corner of a rectangle, leaving us with an $n \times (n-1)$ grid of available red dots to take as the other corner of the rectangle.



Since there are n blue dots, we get an initial total of $n \cdot n \cdot (n-1)$ rectangles using two blue dots and two red dots; however, each rectangle uses two blue dots and so each rectangle has been counted twice. Therefore, there are exactly $\frac{n \cdot n \cdot (n-1)}{2}$ rectangles using two blue dots and two red dots. Due to our observation, this implies that there are also $\frac{n \cdot n \cdot (n-1)}{2}$ rectangles using two green dots and two red dots.

Finally, we must count the number of rectangles using the black dot. However, this is straightforward because the black dot must be the lower left corner of the rectangle, so we simply must choose one of the n^2 red dots as the upper right corner, giving us n^2 rectangles.

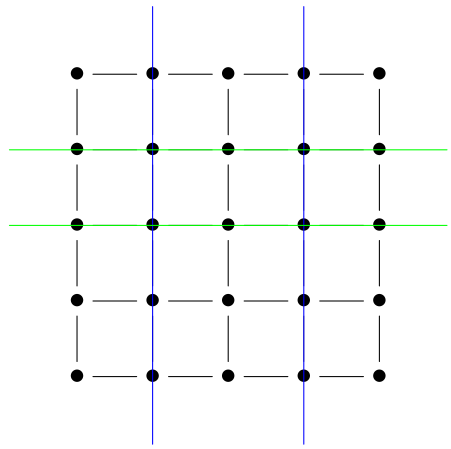
Putting all of this information together, we have determined that

$$\begin{aligned} R_{n+1} &= R_n + \frac{n \cdot n \cdot (n-1)}{2} + \frac{n \cdot n \cdot (n-1)}{2} + n^2 \\ &= R_n + n \cdot n \cdot (n-1) + n^2 \\ &= R_n + n^3 - n^2 + n^2 \\ &= R_n + n^3. \end{aligned}$$

We now have a formula that tells us R_{n+1} if we know R_n. Indeed, $R_{n+1} = R_n + n^3$ gives us a *recurrence relation*. Given that $R_2 = 1 = 1^3$, we have that

$$R_{n+1} = 1^3 + 2^3 + \cdots + n^3.$$

Now, let's find R_{n+1} in a different way:



In this previous picture, we see that if we choose any two columns (blue lines) and choose any two rows (green lines), then we get a unique rectangle bounded by our four choices. Moreover, every rectangle inside the grid can be constructed in this manner. In the $(n+1) \times (n+1)$ grid, there are $n+1$ columns and $n+1$ rows. We let $\binom{n+1}{2}$ represent the number of ways to choose 2 columns (or rows) from the $n+1$ columns (or rows). Therefore, we have come to the conclusion that

$$R_{n+1} = \binom{n+1}{2} \cdot \binom{n+1}{2} = \binom{n+1}{2}^2.$$

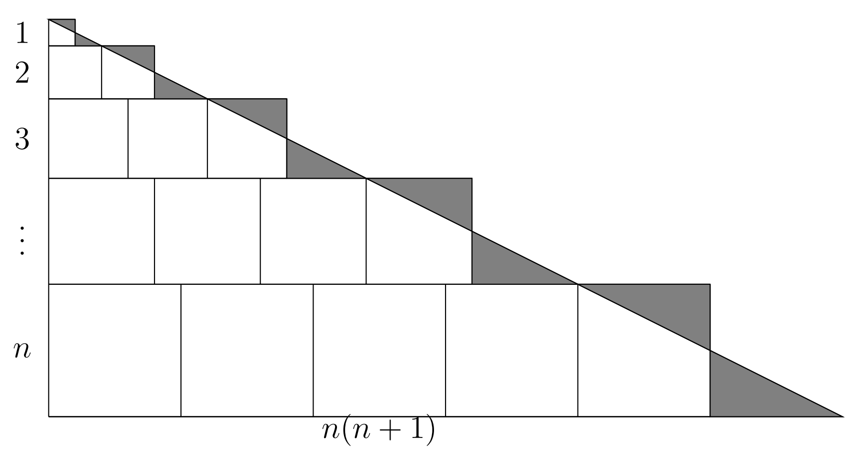
Therefore

$$(1+2+3+\cdots+n)^2 = 1^3 + 2^3 + 3^3 + \cdots n^3 = R_{n+1} = \binom{n+1}{2}^2.$$

Taking square roots gives the result.

21. **Via a linear function**:

Consider the function $f(x) = -\frac{1}{2}x + (1+2+3+\cdots+n)$ pictured below.



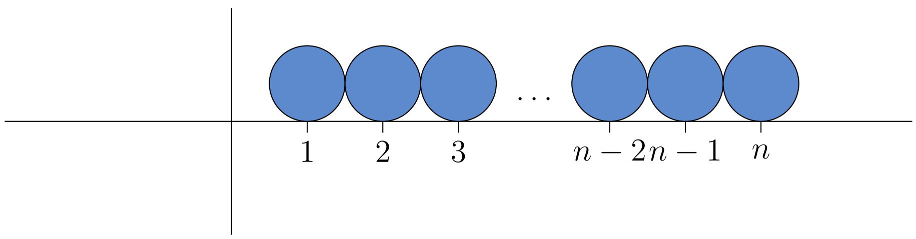
We see from the picture that the x-intercept is $(n(n+1), 0)$, so that $0 = f(n(n+1)) = -\frac{1}{2}(n(n+1)) + (1+2+3+\cdots+n)$. This visual proof is due to Georg Schrage [16].

22. **Via centers of mass**:

Let $\{w_j\}_{j=1}^{n}$ be a set of weights where w_j has mass m_j and sits at coordinates (x_j, y_j) in the real plane. The center of mass of the configuration of weights has coordinates $(\overline{x}, \overline{y})$ where

$$\overline{x} = \frac{\sum_{j=1}^{n} m_j x_j}{\sum_{j=1}^{n} m_j} \text{ and } \overline{y} = \frac{\sum_{j=1}^{n} m_j x_j}{\sum_{j=1}^{n} m_j}.$$

These coordinates are called the x-mean and the y-mean respectively. Now, assume that each weight, w_j, has mass $m_j = 1$ and sits at the coordinate $(j, .5)$ as pictured below.



Since the masses are uniform, we know that the x-mean sits at the midpoint of the masses, which is at $\frac{n+1}{2}$. However, according to the formula above, the x-mean is given by $\frac{\sum_{j=1}^{n} j}{\sum_{j=1}^{n} 1} = \frac{1+2+3+\cdots+n}{n}$. Thus, we have

$$\frac{1+2+3+\cdots+n}{n} = \frac{n+1}{2}$$

and the result follows after multiplying both sides by n.

This result is by Treeby [17].

23. **Via Abel's transformation (summation by parts)**:

The following result is known as Abel's transformation or summation by parts.

Theorem 10. Let $(a_1, a_2, a_3, \ldots)$ and $(b_1, b_2, b_3, \ldots)$ be sequences of positive real numbers and let $n \in \mathbb{N}$. If $B_i := b_1 + b_2 + \cdots + b_i$, then

$$\sum_{k=1}^{n} a_k b_k = B_n a_n - \sum_{k=1}^{n-1} B_k (a_{k+1} - a_k).$$

For a visual proof of this result see [2].

If we let $a = (1, 2, 3, 4, \ldots)$ and $b = (1, 1, 1, 1, \ldots)$, we get $B = (1, 2, 3, 4, \ldots)$ and so $B_k = k$, $a_k = k$ and $b_k = 1$ for all k. Applying Abel's theorem in this case we have

$$\sum_{k=1}^{n} k\cdot 1 = n\cdot n - \sum_{k=1}^{n-1} k((k+1)-k) = n^2+n-\sum_{k=1}^{n} k\cdot(1).$$

or

$$2\sum_{k=1}^{n} k = n^2+n.$$

24. **Using a permutation**:
The following proof is from [12]. Break the square $[0,n-1]\times[0,n-1]$ into two triangles, each of which is given in $\mathbb{R}^2$ by the inequality $x_{\sigma_1}\geq x_{\sigma_2}$ for a permutation $\sigma=\begin{pmatrix}1 & 2\\ \sigma_1 & \sigma_2\end{pmatrix}$ of the set $\{1,2\}$ as in Figure 9. Note that in Figure 9, the red triangle corresponds to $\sigma=1$ and the blue triangle to $\sigma=(12)$. Shift a triangle corresponding to σ by the vector such that its ith component is given by the number of inversions (ji) in σ such that $j<i$. The red triangle stays put ($\sigma=1$) and the blue triangle is shifted by 1. The lattice points of the two shifted triangles cover exactly $n(n+1)$ lattice points. Since each triangle covers $1+2+\cdots+n$ lattice points, we have $2(1+2+\cdots+n)=n(n+1)$.

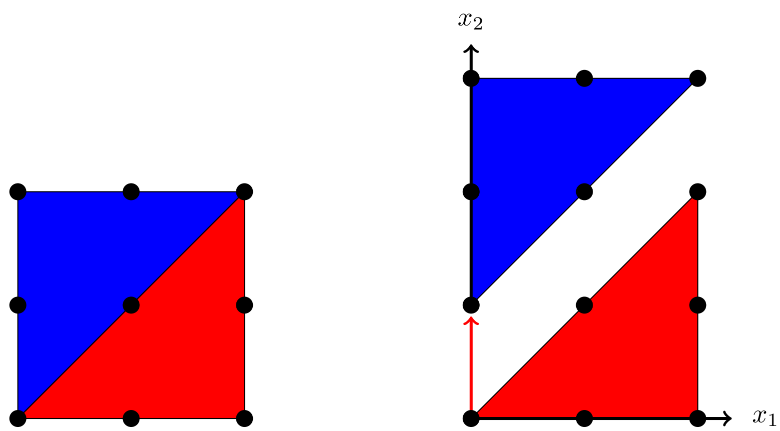


Figure 9: Breaking a square into two triangles.

Remark 11. The proof is essentially the same as Proof 14. However, viewing the proof this way, leads to some nice generalizations. For example, by breaking the $[0,n-1]^3$ cube into six simplices based on permutations, Marko and Litvinov [12] give a nice proof that

$$6\sum_{i=1}^{n}\sum_{j=1}^{i} j = n(n+1)(n+2).$$

25. **Labeling lattice points in a simplex**:
This proof is also from [12]. Let S_n^k denote the k-dimensional simplex given in $\mathbb{R}^k$ by the vertices $V_0=(0,\cdots,0)$ and $V_n^i=(n-1,\ldots,n-1,0,\ldots,0)$, where $i=1,2,\ldots,k$ is such that the entry $n-1$ appears in the first i coordinates of V_n^i. Consider S_n^1, i.e., the points are $V_0=(0)$ and $V_n^1=(n-1)$. The simplex is just the line segment from 0 to $n-1$. It has n lattice points. Now, label each lattice point with two values: $i+1$ and

$n-i$. Each number between 1 and n appears twice as a label. Therefore, the sum of the labels is $2(1+2+\cdots+n)$. But, the sum of the labels at each lattice point is $n+1$ and there are n lattice points, so the total sum is $n(n+1)$. Therefore $1+2+\cdots+n = \frac{n(n+1)}{2}$.

Remark 12. This proof is a complicated way of doing the Gauss trick. However, the idea of giving two labels to each lattice point, has a nice generalization. Namely, Marko and Litvinov assigned three values to lattice points in the simplex S_n^2 to prove for any pair of reals a, b that

$$3\sum_{i=1}^{n}(a\,i^2 + b\,i) = ((2n+1)a + 3b)\,\frac{n(n+1)}{2}.$$

Note that taking $a = 1, b = 0$ reveals the formula for the sum of the first n squares.

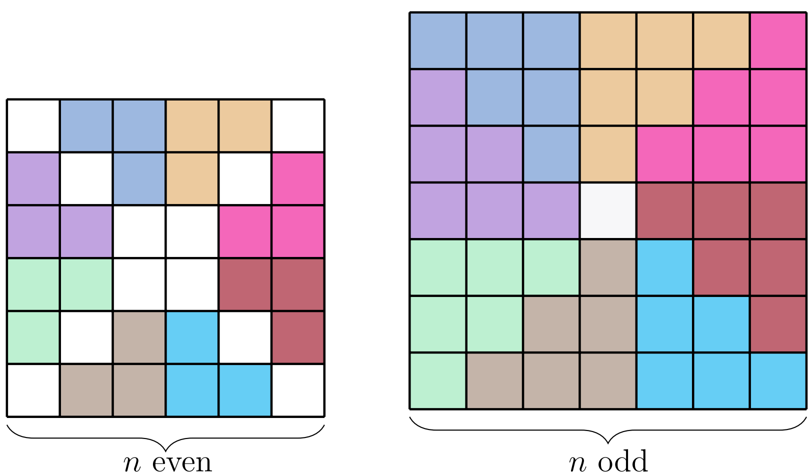
26. **Via squares modulo 8**:

Each perfect square is known to be congruent to 0 or 1 modulo 8 (or 4 or 2). This theorem has a classic visual proof due to Landauer [8, 9].

Theorem 13. For any integer $n \geq 1$,

$$n^2 = \begin{cases} 8t_k + 1 & \text{if } n = 2k+1 \text{ is odd;} \\ 8t_{\ell-1} + 4\ell & \text{if } n = 2\ell \text{ is even.} \end{cases}$$

Proof.



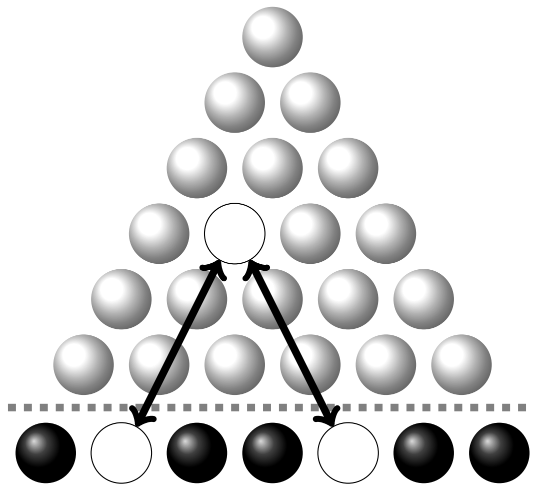
where $t_k = 1 + 2 + \cdots + k$.

□

As a corollary to this theorem, we see that $(2k+1)^2 = 8t_k + 1$ so that $4k^2 + 4k = 8t_k$. This implies the desired result $t_k = \frac{4k^2+4k}{8} = \frac{2k^2+2k}{2}$ for all k (one can also use the even case).

27. **A visual bijection**:

The following visual describes a bijection between the sum $1+2+3+\cdots+n$ and the collections of 2-subsets of the set $\{1,2,3,\ldots,n+1\}$. This visual is essentially a visual proof describing the bijection given in proof 7.



Therefore $1+2+3+\cdots+n = \binom{n+1}{2}$.

28. **Via 2×2 matrices**:

The following proof was suggested by Larson [10] as an exercise. For any $n \in \mathbb{N}$, let $M_n = \begin{bmatrix} 1+2+3+\cdots+n & n \\ n+1 & 2 \end{bmatrix}$. Now, we note that

$$(M_n \cdot E)^T = \left(\begin{bmatrix} 1+2+3+\cdots+n-1 & n \\ n-1 & 2 \end{bmatrix}\right)^T = \begin{bmatrix} 1+2+3+\cdots+n-1 & n-1 \\ n & 2 \end{bmatrix} = M_{n-1}$$

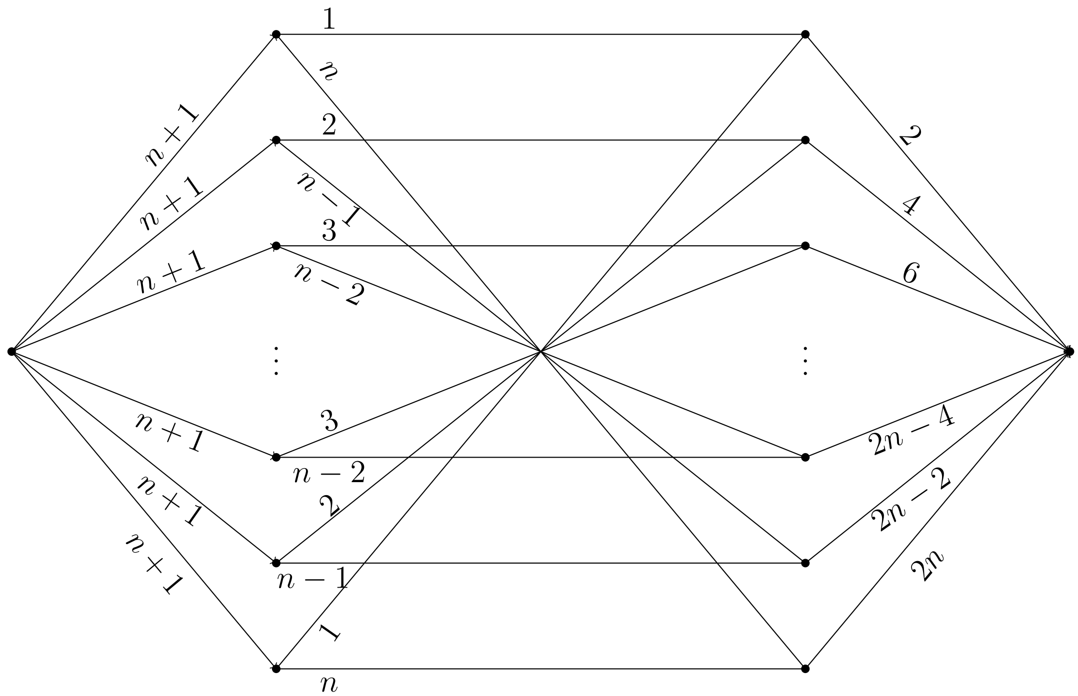
where $E = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$. Note that $\det(E) = 1$, so that $\det(M_{n-1}) = \det(M_n)$ for all n. However, we see that $M_1 = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$, which has determinant 0. Thus

$$(1+2+3+\cdots+n)\cdot 2 - n\cdot(n+1) = \det(M_n) = \det(M_1) = 0$$

and the result follows.

29. **Via water flow diagrams**:

This proof is from [10]. The following figure represents the flow of water from a source S to a terminal point T via a system of aqueducts. Leaving S and entering T, there are n aqueducts. Each edge label represents the rate of flow. The incoming and outgoing rates at T and S respectively must be equal (since no water is lost).



Thus $n \cdot (n+1) = \sum_{i=1}^{n} 2i = 2(1+2+3+\cdots+n)$.

30. **Via simple telescoping**:

First, we note that for all $i \in \mathbb{N}$, we have $i = \dfrac{i^2}{2} + \dfrac{i}{2} - \dfrac{i^2}{2} + \dfrac{i}{2} = \dfrac{i(i+1)}{2} - \dfrac{i(i-1)}{2}$. Then, we have

$$\begin{aligned}
\sum_{i=1}^{n} i &= \sum_{i=1}^{n} \frac{i(i+1)}{2} - \frac{i(i-1)}{2} \\
&= \sum_{i=1}^{n} \frac{i(i+1)}{2} - \sum_{i=1}^{n} \frac{i(i-1)}{2} \\
&= \sum_{i=1}^{n} \frac{i(i+1)}{2} - \sum_{i=0}^{n} \frac{(i+1)i}{2} \\
&= \frac{n(n+1)}{2} + \sum_{i=1}^{n-1} \frac{i(i+1)}{2} - \sum_{i=1}^{n-1} \frac{(i+1)i}{2} \\
&= \frac{n(n+1)}{2}.
\end{aligned}$$

Remark 14. We note that this has basically the same structure as proof 10, without resorting to the fact that sums of consecutive odds yields squares.

31. **Via Pick's Theorem:**

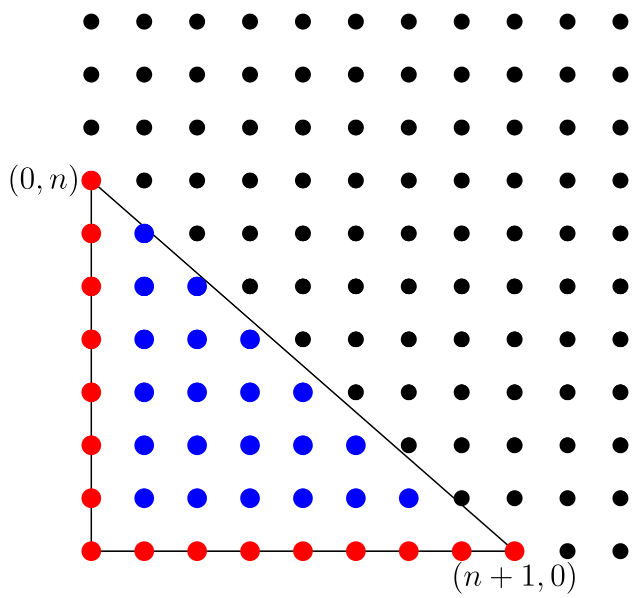
This proof was given as an exercise in [10]. We first recall Pick's Theorem.

Theorem 15 (Pick)**.** Let P be a lattice polygon (a polygon with vertices at integer lattice points). The area A of P is given by

$$A = I + \frac{B}{2} - 1,$$

where I is the number of lattice points lying in the interior of P and B is the number of lattice points lying on the boundary of P.

To apply the theorem, we consider, for any n, the triangle with vertices $(0,0)$, $(0,n)$, and $(n+1,0)$ as pictured below.



There are $1+2+3+\cdots+(n-1)$ interior (blue) dots and there are $(n+1)$ vertical boundary dots, $(n+2)$ horizontal boundary dots, resulting in $(n+1)+(n+2)-1 = 2n+2$ total boundary (red) dots. Moreover, the area of the triangle is $\frac{n(n+1)}{2}$. Thus by Pick's Theorem we have

$$\frac{n(n+1)}{2} = 1+2+3+\cdots+(n-1)+\frac{1}{2}(2n+2)-1 = 1+2+3+\cdots+(n-1)+n,$$

as required.

32. **Via Euler's Theorem**:

This proof was also given as an exercise in [10]. We first recall that Euler's polyhedral formula as stated for planar graphs (graphs that can be drawn in the plane so that no edges cross).

Theorem 16 (Euler)**.** Let G be a finite planar graph with v vertices and e edges that divides the plane up into f faces (regions bounded by edges including the outer infinite region). Then

$$v - e + f = 2.$$

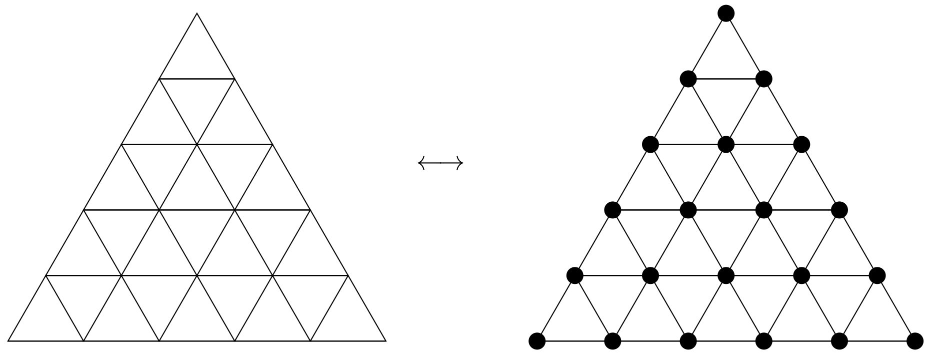


Figure 10: A triangular representation of perfect squares (there are $5^2 = 25$ small triangles) turned into the graph G_n (when $n = 5$).

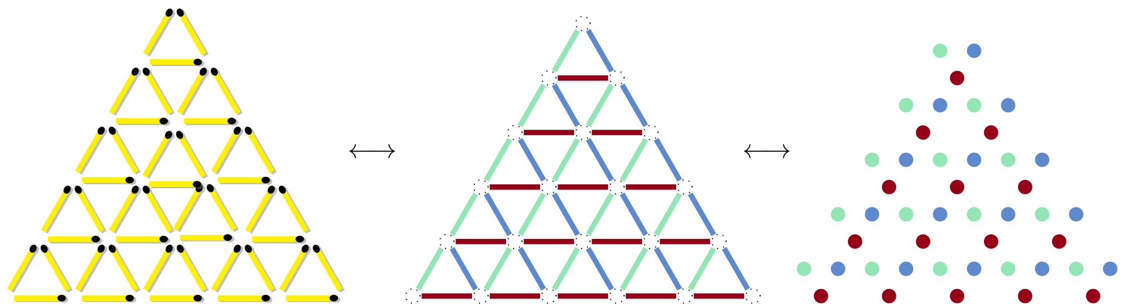


Figure 11: A visual proof showing that G_n has $3t_n$ edges ($3t_n$ represents the matchstick numbers) [4].

Now, recall that we let $t_k = 1 + 2 + 3 + \cdots + k$ for any k.

One classic visual way to represent the perfect squares is with a triangular array of odd rows of triangles as pictured on the left in Figure 10 (see [1]). Then, for any n, let G_n be the corresponding graph pictured on the right in figure 10. The graph G_n is clearly a planar graph containing t_{n+1} vertices. The graph's interpretation as a representation of squares implies that it divides the plane into $n^2 + 1$ faces. Finally, the set of edges of G_n represent a famous collection of numbers known as the matchstick numbers. Figure 11 shows that the matchstick number for the graph G_n is given by $3t_n$, thus G_n has $3t_n$ edges. Hence, applying Euler's formula to the graph G_n, which has $v = t_{n+1}$, $e = 3t_n$ and $f = n^2 + 1$, we get

$$t_{n+1} - 3t_n + n^2 + 1 = 2.$$

After noting that $t_{n+1} = 1 + 2 + 3 + \cdots + n + (n + 1) = t_n + (n + 1)$, we get $t_n + (n + 1) - 3t_n + n^2 + 1 = 2$ so that $-2t_n + n^2 + n = 0$ or $n^2 + n = 2t_n$. The result follows after dividing by 2.

33. **Via Viète's formulas**:

Moreno and García-Caballero [13] provided the following proof using Viète's formulas. Let $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial with $a_n \neq 0$. Label the n roots (possibly complex and not necessarily distinct) of $p(x)$ $r_1, r_2, \ldots,$ and r_n. Then, one of Viète's formulas relating the coefficients of $p(x)$ to the roots of $p(x)$ implies

$$-\left(r_1 + r_2 + \cdots + r_n\right) = \frac{a_{n-1}}{a_n}. \tag{4}$$

Now, for any positive integer n, we define the polynomial $P_n(x)$ by

$$P_n(x) = \prod_{i=1}^{n}(x-i) = (x-1)(x-2)\cdots(x-n).$$

The roots of P_n are clearly $r_i = i$ for $1 \le i \le n$. Furthermore, the midpoint of the roots is $\frac{n+1}{2}$, and since the roots are symmetric about the midpoint, we see that

$$P_n\left(\frac{n+1}{2}+x\right) = (-1)^n P_n\left(\frac{n+1}{2}-x\right).$$

From this description, we define the polynomial Q_n by

$$Q_n = P_n\left(\frac{n+1}{2}+x\right);$$

from above, we can see that when n is even, Q_n is even; likewise, when n is odd, Q_n is odd. Thus, in either case, when $Q_n(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots a_1 x + a_0$ with $a_n \neq 0$, we must have $a_{n-1} = 0$ (or else the polynomial would not be even or odd). Now, if we let s_i denote the ith root of Q_n, then $\frac{n+1}{2} + s_i = r_i = i$ is the corresponding root of P_n. Applying Viète's formula (equation (4)) to Q_n, we see that

$$0 = \frac{0}{a_n} = -(s_1 + s_2 + \cdots + s_n).$$

We use this fact to obtain the following result.

$$(1+2+3+\cdots+n) = \sum_{i=1}^{n} r_i = \sum_{i=1}^{n}\left(\frac{n+1}{2}+s_i\right) = n\cdot\frac{n+1}{2} + \sum_{i=1}^{n} s_i = \frac{n(n+1)}{2} + 0,$$

as required.

34. **Using Cauchy's functional equations**:

Cauchy proved that a function $f : \mathbb{Q} \to \mathbb{Q}$ satisfying $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{Q}$ satisfies that for all $r \in \mathbb{Q}$, $f(r) = f(1)r$.

Let $S(n) = 1 + 2 + \ldots + n$. Then

$$\begin{aligned} S(m+n) &= 1 + 2 + \ldots + m + (m+1) + (m+2) + \ldots + (m+n) \\ &= S(m) + S(n) + mn. \end{aligned}$$

Consider the function $f(n) = S(n) - \frac{1}{2}n^2$, defined on positive integers. Then

$$\begin{aligned} f(m+n) &= S(m+n) - \frac{1}{2}(m+n)^2 \\ &= S(m) + S(n) + mn - \frac{m^2}{2} - \frac{n^2}{2} - mn \\ &= f(m) + f(n). \end{aligned}$$

Therefore, $f(n)$ is of the Cauchy type. Therefore $f(n) = f(1)n$. Hence,

$$S(n) - \frac{n^2}{2} = f(n) = f(1)n = \frac{n}{2}.$$

It follows that $S(n) = \frac{n(n+1)}{2}$.

Remark 17. This proof was shared with the second author by José Luis Cereceda. The proof and generalizations to higher powers appear in [15].

35. **Using the Stolz-Cesàro Theorem**:

The following proof, due to Kung [7], was communicated to us by José Luis Cereceda.

Theorem 18 (Stolz-Cesàro Theorem)**.** : Let $\{a_n\}$ and $\{b_n\}$ be two sequences of real numbers. If b_n is positive, strictly increasing, unbounded, and

$$\lim_{n\to\infty} \frac{a_n - a_{n-1}}{b_n - b_{n-1}} = \ell,$$

then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \ell.$$

Now, consider the sequences $a_n = 1 + 2 + \ldots + n$, and $b_n = n^2$. Note that

$$\lim_{n\to\infty} \frac{a_n - a_{n-1}}{b_n - b_{n-1}} = \lim_{n\to\infty} \frac{n}{2n-1} = \frac{1}{2}.$$

Therefore, by Stolz-Cesàro,

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1 + 2 + \ldots + n}{n^2} = \frac{1}{2}.$$

Now, consider the sequence $s_n = a_n - \frac{1}{2}n^2$. Then

$$s_n - s_{n-1} = n - \frac{n^2}{2} + \frac{(n-1)^2}{2} = n + \frac{-2n+1}{2} = \frac{1}{2}.$$

Therefore,

$$\lim_{n\to\infty} \frac{s_n - s_{n-1}}{n - (n-1)} = \lim_{n\to\infty} \frac{\frac{1}{2}}{1} = \frac{1}{2}.$$

By Stolz-Cesàro,

$$\lim_{n\to\infty} \frac{s_n}{n} = \frac{1}{2}.$$

Now let $r_n = a_n - \frac{n^2}{2} - \frac{n}{2}$, then

$$r_n - r_{n-1} = \frac{1}{2} - \frac{n}{2} + \frac{n-1}{2} = 0.$$

Therefore, for any increasing sequence b_n that goes to infinity, we have

$$\lim_{n\to\infty} \frac{r_n - r_{n-1}}{b_n - b_{n-1}} = \lim_{n\to\infty} \frac{0}{b_n - b_{n-1}} = 0,$$

which implies

$$\lim_{n\to\infty} \frac{r_n}{b_n} = 0.$$

That means that r_n is bounded by a constant. This suggests that $\left| a_n - \frac{n^2}{2} - \frac{n}{2} \right| \leq C$, for some constant C. However, since $r_n - r_{n-1} = 0$, that means the sequence a_n and the sequence $c_n = n^2/2 + n/2$ both satisfy the same recurrence, namely $a_n = n + a_{n-1}$ and $c_n = n + c_{n-1}$. Since $a_0 = c_0 = 0$, then they must be the same sequence. Therefore $a_n = n^2/2 + n/2$.

Remark 19. This technique can be generalized to higher powers as done in [7].

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