The least quadratic non-residue and related problems

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LAKE FOREST COLLEGE

Squares

Consider the sequence

$$2, 5, 8, 11, \ldots$$

Can it contain any squares?

- No, because $x^2 \not\equiv 2 \bmod 3$ for any x.

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Quadratic residues and non-residues

Let n be a positive integer. For $q \in \{0, 1, 2, \ldots, n-1\}$, we call q a quadratic residue $\bmod n$ if there exists an integer x such that $x^2 \equiv q \pmod{n}$. Otherwise we call q a quadratic non-residue.

- For $n = 3$, the quadratic residues are $\{0, 1\}$ and the quadratic non-residue is 2.
- For $n = 5$, the quadratic residues are $\{0, 1, 4\}$ and the quadratic non-residues are $\{2, 3\}$.
- For $n = 7$, the quadratic residues are $\{0, 1, 2, 4\}$ and the quadratic non-residues are $\{3, 5, 6\}$.
- For $n = p$, an odd prime, there are $\frac{p+1}{2}$ quadratic residues and $\frac{p-1}{2}$ quadratic non-residues.

Least quadratic non-residue

How big can the least quadratic non-residue be?
Let $g(p)$ be the least quadratic non-residue modulo p.

p	Least quadratic non-residue
3	2
5	2
7	3
11	2
13	2
17	3
19	2
23	5
29	2
31	3

p	Least quadratic non-residue
7	3
23	5
71	7
311	11
479	13
1559	17
5711	19
10559	23
18191	29
31391	31
422231	37
701399	41
366791	43
3818929	47

Heuristics

Let $g(p)$ be the least quadratic non-residue $\bmod p$. Let p_i be the i-th prime, i.e, $p_1 = 2, p_2 = 3, \ldots$.

- $\#\{p \leq x \,|\, g(p) = 2\} \approx \frac{\pi(x)}{2}$.
- $\#\{p \leq x \,|\, g(p) = 3\} \approx \frac{\pi(x)}{4}$.
- $\#\{p \leq x \,|\, g(p) = p_k\} \approx \frac{\pi(x)}{2^k}$.
- If $k = \log \pi(x)/\log 2$ you would expect only one prime satisfying $g(p) = p_k$.
- Choosing $k \approx C \log x$, since $p_k \sim k \log k$ we have $g(x) \leq C \log x \log\log x$.

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Theorems on the least quadratic non-residue mod p

Let $g(p)$ be the least quadratic non-residue mod p. Our conjecture is

$$g(p) = O(\log p \log \log p).$$

- Under GRH, Bach showed $g(p) \leq 2\log^2 p$. Soundararajan, Lamzouri and Li improved this to $g(p) \leq \log^2 p$.
- Unconditionally, Burgess showed $g(p) \ll_\epsilon p^{\frac{1}{4\sqrt{e}}+\epsilon}$.
- $\frac{1}{4\sqrt{e}} \approx 0.151633$.
- In the lower bound direction, Graham and Ringrose proved that there are infinitely many p satisfying $g(p) \gg \log p \log \log \log p$, that is

$$g(p) = \Omega(\log p \log \log \log p).$$

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Explicit estimates on the least quadratic non-residue mod p

Norton showed

$$g(p) \leq \begin{cases} 3.9p^{1/4}\log p & \text{if } p \equiv 1 \pmod 4, \\ 4.7p^{1/4}\log p & \text{if } p \equiv 3 \pmod 4. \end{cases}$$

Theorem (ET 2015)

Let $p > 3$ be a prime. Let $g(p)$ be the least quadratic non-residue mod p. Then

$$g(p) \leq \begin{cases} 0.9p^{1/4}\log p & \text{if } p \equiv 1 \pmod 4, \\ 1.1p^{1/4}\log p & \text{if } p \equiv 3 \pmod 4. \end{cases}$$

Explicit estimates on the least quadratic non-residue $\bmod p$

Norton showed

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Theorem (Burgess 1962)

Let $g(p)$ be the least quadratic non-residue mod p. *Let $\varepsilon > 0$. There exists p_0 such that for all primes $p \geq p_0$ we have $g(p) < p^{\frac{1}{4\sqrt{e}}+\varepsilon}$.*

Theorem (ET 2015)

Let $g(p)$ be the least quadratic non-residue mod p. *Let p be a prime greater than 10^{4732}, then $g(p) < p^{1/6}$.*

Consecutive quadratic residues or non-quadratic residues

Let $H(p)$ be the largest string of consecutive nonzero quadratic residues or quadratic non-residues modulo p.
For example, with $p = 7$ we have that the nonzero quadratic residues are $\{1, 2, 4\}$ and the quadratic non-residues are $\{3, 5, 6\}$. Therefore $H(7) = 2$.

p	$H(p)$
11	3
13	4
17	3
19	4
23	4
29	4
31	4

Burgess proved in 1963 that $H(p) \leq Cp^{1/4} \log p$.

Mathematician	Year	C	Restriction
Norton*	1973	2.5	$p > e^{15}$
Norton*	1973	4.1	None
Preobrazhenskaya	2009	$1.85\ldots + o(1)$	Not explicit
McGown	2012	7.06	$p > 5 \cdot 10^{18}$
McGown	2012	7	$p > 5 \cdot 10^{55}$
ET	2012	$1.495\ldots + o(1)$	Not explicit
ET	2012	1.55	$p > 10^{13}$
ET	2012	3.64	None

*Norton didn't provide a proof for his claim.

Quadratic fields and inert primes

- Let d be a squarefree integer.
- Then $\mathbb{Q}(\sqrt{d})$ is a quadratic field.
- A prime $p \in \mathbb{Z}$ is inert if it remains prime when it is lifted to the quadratic field.
- For example $\mathbb{Q}(\sqrt{-1}) = \mathbb{Q}(i) = \{a + bi \mid a, b \in \mathbb{Q}\}$. In this field, the inert primes are the primes $p \equiv 3 \pmod 4$.
- Note that 5 is not prime in $\mathbb{Q}(i)$ because $(1 + 2i)(1 - 2i) = 5$. Similarly any prime $p \equiv 1 \pmod 4$ is not prime in $\mathbb{Q}(i)$ since p can be written as $a^2 + b^2$ for some $a, b \in \mathbb{Z}$ and hence $p = (a + bi)(a - bi)$.

Characterization of inert primes in quadratic fields

- The discriminant D of a quadratic field $\mathbb{Q}(\sqrt{d})$ is d if $d \equiv 1 \pmod 4$ and $4d$ otherwise.
- A prime p is inert in $\mathbb{Q}(\sqrt{d})$ if and only if the Kronecker symbol $(D/p) = -1$.
- The Kronecker symbol is a generalization of the Legendre symbol

$$\left(\frac{a}{p}\right) = \begin{cases} 1 & \text{if } a \text{ is a square } \bmod p, \\ -1 & \text{if } a \text{ is a quadratic non-residue } \bmod p, \\ 0 & \text{if } p \mid a. \end{cases}$$

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The least inert prime in a real quadratic field

Theorem (Granville, Mollin and Williams, 2000)

For any positive fundamental discriminant $D > 3705$, there is always at least one prime $p \leq \sqrt{D}/2$ such that the Kronecker symbol $(D/p) = -1$.

Theorem (ET, 2010)

For any positive fundamental discriminant $D > 1596$, there is always at least one prime $p \leq D^{0.45}$ such that the Kronecker symbol $(D/p) = -1$.

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Elements of the Proof

- Use a computer to check the “small” cases. Granville, Mollin and Williams used the Manitoba Scalable Sieving Unit.
- Use analytic techniques to prove it for the “infinite case”, i.e. the very large D. The tool used by Granville et al. was the Pólya–Vinogradov inequality. I used a “smoothed” version of it.
- Use Pólya–Vinogradov plus a bit of clever computing to fill in the gap.

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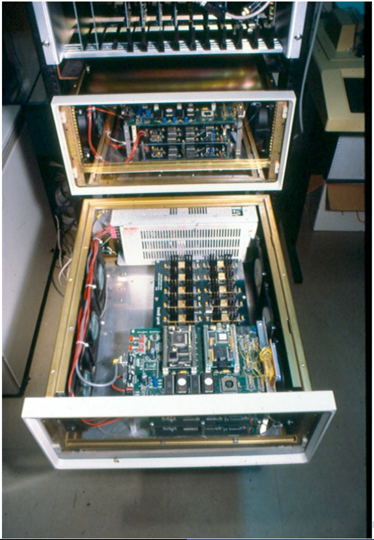
Manitoba Scalable Sieving Unit



Legendre Symbol

$$Let \left(\frac{a}{p}\right) = \begin{cases} 0 & , \quad \text{if } a \equiv 0 \mod p, \\ 1 & , \quad \text{if } a \text{ is a square } \mod p \\ -1 & , \quad \text{if } a \text{ is a quadratic non-residue } \mod p. \end{cases}$$

$\left(\frac{a}{p}\right)$ has the following important properties:

- $\left(\frac{a}{p}\right) = \left(\frac{a+p}{p}\right)$ for all a.
- $\left(\frac{a}{p}\right)\left(\frac{b}{p}\right) = \left(\frac{ab}{p}\right)$ for all a, b.
- $\left(\frac{a}{p}\right) \neq 0$ if and only if $\gcd(a, p) = 1$.

Dirichlet Character

Let n be a positive integer.
$\chi : \mathbb{Z} \to \mathbb{C}$ is a Dirichlet character mod n if the following three conditions are satisfied:

- $\chi(a + n) = \chi(a)$ for all $a \in \mathbb{Z}$.
- $\chi(ab) = \chi(a)\chi(b)$ for all $a, b \in \mathbb{Z}$.
- $\chi(a) \neq 0$ if and only if $\gcd(a, n) = 1$.

The Legendre symbol is an example of a Dirichlet character.

A simple but powerful idea

Let $g(p) = m$ be the least quadratic non-residue modulo p. Suppose $\chi(a) = \left(\frac{a}{p}\right)$ Then $\chi(n) = 1$ for $n = 1, 2, 3, ..., m - 1$ and $\chi(m) = -1$. Therefore

$$\sum_{i=1}^{m} \chi(i) = m - 2 < m,$$

and

$$\sum_{i=1}^{k} \chi(i) = k \text{ for all } k < m.$$

Therefore bounding $\sum_{i=1}^{n} \chi(i)$ can give an upper bound for $g(p)$.

Pólya–Vinogradov

Let χ be a Dirichlet character to the modulus $q > 1$. Let

$$S(\chi) = \max_{M,N} \left| \sum_{n=M+1}^{M+N} \chi(n) \right|$$

The Pólya–Vinogradov inequality (1918) states that there exists an absolute universal constant c such that for any Dirichlet character $S(\chi) \leq c\sqrt{q}\log q$.

Under GRH, Montgomery and Vaughan showed that $S(\chi) \ll \sqrt{q}\log\log q$.

Paley showed in 1932 that there are infinitely many quadratic characters such that $S(\chi) \gg \sqrt{q}\log\log q$.

Vinogradov's Trick: Showing $g(p) \ll p^{\frac{1}{2\sqrt{e}}+\varepsilon}$

- Suppose $\sum_{n \leq x} \chi(n) = o(x)$.
- Let $y = x^{1/\sqrt{e}+\delta}$ for some $\delta > 0$. So $\log\log x - \log\log y = \log(1/\sqrt{e}+\delta) < 1/2$
- Suppose $g(p) > y$.

$$\sum_{n \leq x} \chi(n) = \sum_{n \leq x} 1 - 2 \sum_{\substack{y < q \leq x \\ \chi(q) = -1}} \sum_{n \leq \frac{x}{q}} 1,$$

where the sum ranges over q prime. Therefore we have

$$\sum_{n \leq x} \chi(n) \geq \lfloor x \rfloor - 2 \sum_{y < q \leq x} \left\lfloor \frac{x}{q} \right\rfloor \geq x - 1 - 2x \sum_{y < q \leq x} \frac{1}{q} - 2 \sum_{y < q \leq x} 1.$$

It took almost 50 years before the next breakthrough. It came from the following theorem of Burgess:

Theorem (Burgess, 1962)

Let χ be a primitive character $\bmod q$, where $q > 1$, r is a positive integer and $\epsilon > 0$ is a real number. Then

$$|S_\chi(M,N)| = \left| \sum_{M<n\leq M+N} \chi(n) \right| \ll N^{1-\frac{1}{r}} q^{\frac{r+1}{4r^2}+\epsilon}$$

for $r = 1, 2, 3$ and for any $r \geq 1$ if q is cubefree, the implied constant depending only on ϵ and r.

Consider

$$\left|\sum_{n\leq N}\chi(n)\right|.$$

By Burgess

$$\left|\sum_{n\leq N}\chi(n)\right| \ll N^{1-\frac{1}{r}}q^{\frac{r+1}{4r^2}+\epsilon}.$$

However, if $\chi(n)=1$ for all $n\leq N$, then

$$N\leq\left|\sum_{n\leq N}\chi(n)\right| \ll N^{1-\frac{1}{r}}q^{\frac{r+1}{4r^2}+\epsilon},$$

so

$$N^{\frac{1}{r}} \ll q^{\frac{r+1}{4r^2}+\epsilon}.$$

Hence

$$N \ll q^{\frac{1}{4}+\frac{1}{4r}+\epsilon r}.$$

Now we know why

$$g(p) \ll p^{\frac{1}{4\sqrt{e}}+\varepsilon},$$

but how do we go from there to be able to figure out the theorem:

Theorem (ET 2015)

Let $g(p)$ be the least quadratic non-residue mod *p. Let p be a prime greater than 10^{4732}, then $g(p) < p^{1/6}$.*

Explicit Burgess

Theorem (Iwaniec-Kowalski-Friedlander)

Let χ be a non-principal Dirichlet character mod p (a prime). Let M and N be non-negative integers with $N \geq 1$ and let $r \geq 2$, then

$$|S_\chi(M, N)| \leq 30 \cdot N^{1-\frac{1}{r}} p^{\frac{r+1}{4r^2}} (\log p)^{\frac{1}{r}}.$$

Theorem (ET)

Let p be a prime. Let χ be a non-principal Dirichlet character mod *p. Let M and N be non-negative integers with $N \geq 1$ and let r be a positive integer. Then for $p \geq 10^7$, we have*

$$|S_\chi(M, N)| \leq 2.74 N^{1-\frac{1}{r}} p^{\frac{r+1}{4r^2}} (\log p)^{\frac{1}{r}}.$$

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Some Applications of the Explicit Estimates

- Booker (2006) computed the class number of a 32-digit discriminant using an explicit estimate of a character sum.
- McGown proved (2012) that there is no norm-Euclidean cubic field with discriminant $> 10^{140}$. Recently (2017) improved to no Norm-Euclidean fields with discriminant $> 10^{100}$
- Levin, Pomerance and Soundararajan proved a conjecture of Brizolis that for every prime $p > 3$ there is a primitive root g and an integer $x \in [1, p-1]$ with $\log_g x = x$, that is, $g^x \equiv x \pmod{p}$.
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Key Inequality to prove Burgess Inequality

Theorem (Burgess, Booker, ET)

Let h and w be positive integers. Let χ be a primitive Dirichlet character mod *p, then*

$$\sum_{m=1}^{p}\left|\sum_{l=0}^{h-1}\chi(m+l)\right|^{2w} < (2w-1)!!p\,h^{w} + (2w-1)\sqrt{p}\,h^{2w}.$$

Sketch of a Proof

- $$\sum_{m=1}^{p}\left|\sum_{l=0}^{h-1}\chi(m+l)\right|^{2w} = \sum_{l_1,l_2,\ldots,l_{2w}}\sum_{x \bmod p}\chi(q(x)),$$

  where

  $$q(x) = \frac{(x+l_1)(x+l_2)\ldots(x+l_w)}{(x+l_{w+1})(x+l_{w+2})\ldots(x+l_{2w})}.$$

- If $q(x)$ is not a k-th power (where k is the order of χ), then

  $$\left|\sum_{x \bmod p}\chi(q(x))\right| \leq (r-1)\sqrt{p},$$

  where r is the number of distinct roots of $q(x)$.

Sketch of a Proof

- $$\sum_{m=1}^{p}\left|\sum_{l=0}^{h-1}\chi(m+l)\right|^{2w} = \sum_{l_1,l_2,\ldots,l_{2w}}\sum_{x \bmod p}\chi(q(x)),$$
  where
  $$q(x) = \frac{(x+l_1)(x+l_2)\ldots(x+l_w)}{(x+l_{w+1})(x+l_{w+2})\ldots(x+l_{2w})}.$$
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Applications

Theorem (ET 2015)

Let $p > 3$ be a prime and k be a positive integer that divides $p-1$. Let $g(p,k)$ be the least k-th power non-residue $\bmod p$. Then

$$g(p,k) \leq \begin{cases} 1.1p^{1/4}\log p & \text{if } p \equiv 3 \bmod 4 \text{ and } k = 2, \\ 0.9p^{1/4}\log p & \text{otherwise.} \end{cases}$$

Theorem (ET 2012)

If χ is any non-principal Dirichlet character to the prime modulus p which is constant on $(N, N+H]$, then

$$H \leq \begin{cases} 3.64p^{1/4}\log p, & \text{for all odd } p, \\ 1.55p^{1/4}\log p, & \text{for } p \geq 2.5\cdot 10^9. \end{cases}$$

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Thank you!