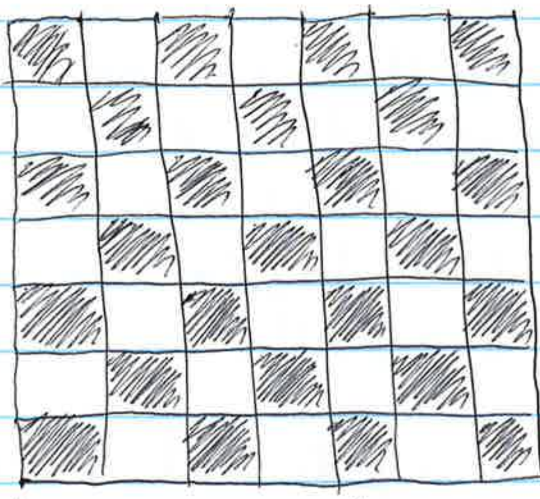
MATH 28 9/11/2012

HOMEWORK #1

SOLUTIONS

(1) Place a knight on each square of a 7x7 chessboard. Is it possible for each knight to simultaneously make a legal move?

SOLUTION: No. Suppose it were possible. Coloring the board in black and white alternatively as follows



has 25 black squares and 24 white squares.

A legal knight move takes a knight from a black square to a white square (or viceversa).
Hence the 25 knights standing in black squares should end up in 25 white squares, but there's only 24. Hence it isn't possible for all knights to make legal moves.

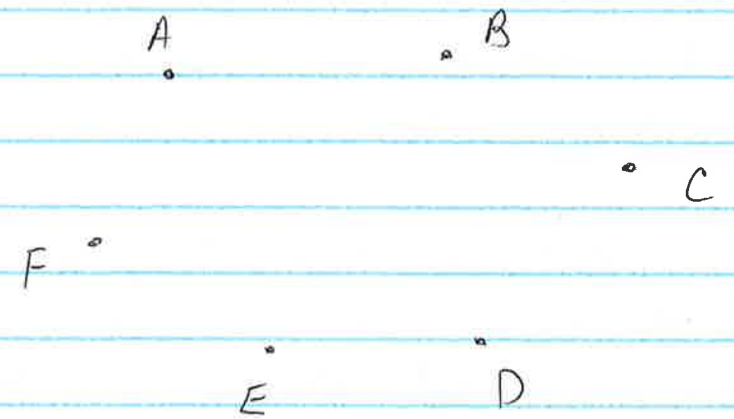
(2) In a party with 45 people, among any set of four there is at least one person who knows each of the other three. There are three people who are not mutually acquainted with each other. Prove that the other 42 people know everyone at the party.

SOLUTION: Suppose there is one person among those 42 who doesn't know everybody. Let's call him F. Let A, B and C be the three people who are not mutually acquainted. Then by considering the group of four {A, B, C, F} since A, B, C are not mutually acquainted and at least one of the four knows the rest, we conclude F knows A, B and C. Since F doesn't know everybody and he knows A, B, C, then there must be a person D ≠ A, B, C such that F doesn't know D. But now consider {A, B, D, F}. Since A doesn't know B and D doesn't know F then none of the four knows everybody, contradicting our assumption. Therefore everybody (of the group of 42) knows everybody.

③ Show that if you have a party with at least 6 people, there are three people who are all either mutual acquaintances or mutual strangers.

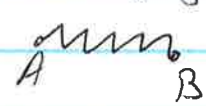
SOLUTION: Consider 6 people A, B, C, D, E, F. Assume the statement is false.

Represent the people by points.



If A knows B we draw a line segment

If A doesn't know B we draw a squigly line

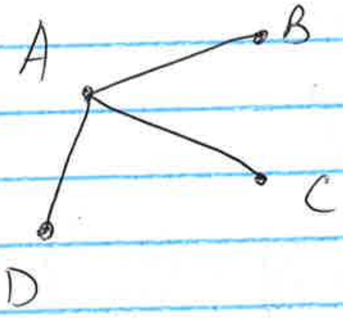


Similarly for all pairs of people.

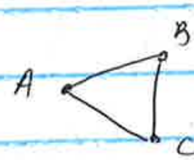
Consider A.

There are 5 possible acquaintences. So either you have at least 3 acquaintances or at least 3 strangers. Both cases are symmetric. So assume A has 3 "acquaintances". And reorder the names if necessary to call the friends B, C and D.

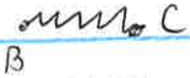
Now it looks like this:



Now if 

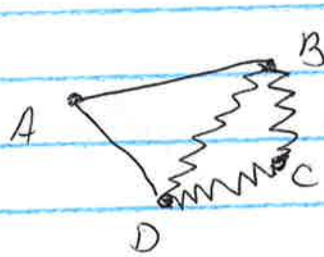


so A, B, C are mutual acquaintances.

Therefore 

similarly for 

and 

But now we have 

, so 

meaning B, C, D are mutual strangers. This contradicts our assumption. Therefore either 3 mutual acquaintances or 3 mutual strangers must be in the party.

(4) Show, for all integers $n \geq 1$, that

$$1^3 + 2^3 + \ldots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$$

Proof: Let's prove it by induction.

For $n=1$ we have $1 = \left(\frac{1(2)}{2}\right)^2 = 1^2 = 1$ ✓

Assume it is true for $n=k$, i.e., $1^3 + \ldots + k^3 = \left(\frac{k(k+1)}{2}\right)^2$

Let's show the statement is true for $n=k+1$.

$$(1^3 + 2^3 + \ldots + k^3) + (k+1)^3 = \left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3$$

$$= \frac{(k+1)^2}{4}\left(k^2 + 4(k+1)\right) = \frac{(k+1)^2}{4}\left(k^2 + 4k + 4\right)$$

$$= \left(\frac{k+1}{2}\right)^2 (k+2)^2 = \left(\frac{(k+1)(k+2)}{2}\right)^2$$

which is what we wanted to prove $\boxtimes$

(5) Calculate $\frac{1}{1 \times 3} + \frac{1}{3 \times 5} + \ldots + \frac{1}{999 \times 1001}$.

Solution: Let's try to find a pattern:

$$\frac{1}{1 \times 3} = \frac{1}{3}$$

$$\frac{1}{1 \times 3} + \frac{1}{3 \times 5} = \frac{1}{3} + \frac{1}{15} = \frac{6}{15} = \frac{2}{5}$$

$$\frac{1}{1 \times 3} + \frac{1}{3 \times 5} + \frac{1}{5 \times 7} = \frac{2}{5} + \frac{1}{35} = \frac{14+1}{35} = \frac{3}{7}$$

$\frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \ldots$ the pattern seems to be $\frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{4}{9}, \frac{5}{11}, \ldots$

So let's try to prove

$$\frac{1}{1\times 3} + \frac{1}{3\times 5} + \cdots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$$ by induction.

For $n=1$ we have

LHS: $\frac{1}{(2(1)-1)(2(1)+1)} = \frac{1}{(1)(3)} = \frac{1}{3}$

RHS: $\frac{1}{2(1)+1} = \frac{1}{3}$

So the left hand side equals the right hand side. So the base case is proven.

Now assume

$$\frac{1}{1\times 3} + \cdots + \frac{1}{(2k-1)(2k+1)} = \frac{k}{2k+1}$$

and let's show

$$\frac{1}{1\times 3} + \cdots + \frac{1}{(2k-1)(2k+1)} + \frac{1}{(2k+1)(2k+3)} = \frac{k+1}{2k+3}.$$

$$\left(\frac{1}{1\times 3} + \cdots + \frac{1}{(2k-1)(2k+1)}\right) + \frac{1}{(2k+1)(2k+3)} = \frac{k}{2k+1} + \frac{1}{(2k+1)(2k+3)} = \frac{k(2k+3)+1}{(2k+1)(2k+3)}$$

$$= \frac{2k^2+3k+1}{(2k+1)(2k+3)} = \frac{(2k+1)(k+1)}{(2k+1)(2k+3)} = \frac{k+1}{2k+3}$$

Therefore we proved the inductive step and hence

$$\frac{1}{1\times 3} + \cdots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$$

Therefore $\frac{1}{1\times 3} + \cdots + \frac{1}{999\times 1001} = \boxed{\frac{500}{1001}}$.

Another way to calculate the sum is the following.

$$\frac{1}{1\times 3}+\frac{1}{3\times 5}+\dots+\frac{1}{999\times 1001} = \frac{1}{2}\left(1-\frac{1}{3}\right)+\frac{1}{2}\left(\frac{1}{3}-\frac{1}{5}\right)+\dots+\frac{1}{2}\left(\frac{1}{999}-\frac{1}{1001}\right)$$

$$= \frac{1}{2}\left(1-\frac{1}{3}+\frac{1}{3}-\frac{1}{5}+\frac{1}{5}-\dots-\frac{1}{999}+\frac{1}{999}-\frac{1}{1001}\right)$$

$$= \frac{1}{2}\left(1-\frac{1}{1001}\right) = \frac{1}{2}-\frac{1}{1001} = \boxed{\frac{500}{1001}}$$

(6) Show that the product of three consecutive integers is a multiple of 6.

SOLUTION: Let's show that for all $n\geq 1$, $(n)(n+1)(n+2)$ is a multiple of 6. Let's do it by induction.

If $n=1$ then $n(n+1)(n+2) = (1)(2)(3) = 6$ which is indeed a multiple of 6.

If $n=2$, $(2)(3)(4) = 24 = (4)(6)$ ✓

If $n=3$, $(3)(4)(5) = 60 = (10)(6)$ ✓

If $n=4$, $4(5)(6) = 6(20)$ ✓

If $n=5$, $(5)(6)(7) = 6(35)$ ✓

If $n=6$, $(6)(7)(8) = (56)(6)$ ✓

Now assume the statement is true for $n=k$, i.e. $k(k+1)(k+2) = 6l$

Let's show it for $n = k+6$.

$$(k+6)(k+7)(k+8) = (k+6)((k+1)+6)((k+2)+6)$$

$$= k(k+1)(k+2) + 6\left((k+1)(k+2)+k(k+2)+k(k+1)+36\right) + 36(k+k+1+k+2)$$

$$= 6l + 6m + 6t = 6(l+m+t)$$ a multiple of 6.

(note: $m = (k+1)(k+2)+k(k+2)+k(k+1)+36$ and $t = 6(k+k+1+k+2)$)

Since we proved the statement for $n=1,2,3,4,5,6$ and $k \Rightarrow k+6$

then $1 \to 7 \to 13 \to 19 \to \ldots$

$2 \to 8 \to 14 \to 20 \to \ldots$

$3 \to 9 \to 15 \to 21 \to \ldots$ so it's true for all n. $\boxtimes$

$4 \to 10 \to 16 \to 22 \to \ldots$

$5 \to 11 \to 17 \to 23 \to \ldots$

$6 \to 12 \to 18 \to 24 \to \ldots$

There are other easier non-inductive proofs, but I wanted to illustrate the induction proof. If the way it is written is not satisfactory, you can do it by strong induction, using $n=k-5$ to imply $n=k+1$.

(7) The following statement is obviously false: "In any list of n integers, all of them are equal". Find the mistake in the following "induction" proof:

"For $n=1$, the statement is true since the list has only one number. Now suppose the statement is true for any list of k integers. Consider an arbitrary list of $k+1$ integers $\{a_1, a_2, \ldots, a_{k+1}\}$. By the inductive hypothesis applied to the lists $\{a_1, \ldots, a_k\}$ and $\{a_2, \ldots, a_{k+1}\}$ we have

$$a_1 = a_2 = \ldots = a_k$$

and $$a_2 = a_3 = \ldots = a_{k+1}.$$

Therefore $a_1 = a_2 = \ldots = a_{k+1}$, proving that all elements in the list are equal."

SOLUTION: $a_1 = a_2 = \ldots = a_k$ together with $a_2 = a_3 = \ldots = a_{k+1}$ is only true if $k \geq 2$ because if $k=1$

we would have a_1 and a_2 but no connecting equality. If $k=2$ we have $a_1=a_2$ and $a_2=a_3$ which implies $a_1=a_2=a_3$, if $k=3$ we have $a_1=a_2=a_3$ and $a_2=a_3=a_4$ which implies $a_1=a_2=a_3=a_4$, but when $k=1$ the equality doesn't happen.

Since out base case is $n=1$, the inductive step fails to connect $n=1$ with $n=2$ and hence the induction proof is flawed. ☒

Note: If the base case was $n=2$, then the proof would work.