1.1.**12** $\begin{bmatrix} x - 2y & = 3 \\ 2x - 4y & = 6 \end{bmatrix} \begin{matrix} \\ -2(I) \end{matrix} \quad \rightarrow \begin{bmatrix} x - 2y & = 3 \\ 0 & = 0 \end{bmatrix}$

This system has infinitely many solutions: If we choose $y = t$, an arbitrary real number, then the equation $x - 2y = 3$ gives us $x = 3 + 2y = 3 + 2t$. Therefore the general solution is $(x, y) = (3 + 2t, t)$, where t is an arbitrary real number. (See Figure 1.2.)

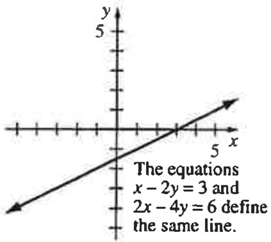

Figure 1.2: for Problem 1.1.12.

$$\begin{bmatrix} x & = -5a+2b \\ y & = 3a-b \end{bmatrix}, \text{ so that } (x,y) = (-5a+2b,\ 3a-b).$$

1.1.**18** $\begin{bmatrix} x+2y+3z & = a \\ x+3y+8z & = b \\ x+2y+2z & = c \end{bmatrix} \begin{matrix} \\ -I \\ -I \end{matrix} \rightarrow \begin{bmatrix} x+2y+3z & = a \\ y+5z & = -a+b \\ -z & = -a+c \end{bmatrix} \begin{matrix} -2(II) \\ \\ \\ \end{matrix} \rightarrow$

$$\begin{bmatrix} x-7z & = 3a-2b \\ y+5z & = -a+b \\ -z & = -a+c \end{bmatrix} \begin{matrix} \\ \\ \div(-1) \end{matrix} \rightarrow \begin{bmatrix} x-7z & = 3a-2b \\ y+5z & = -a+b \\ z & = a-c \end{bmatrix} \begin{matrix} +7(III) \\ -5(III) \\ \\ \end{matrix} \rightarrow \begin{bmatrix} x & = 10a-2b-7c \\ y & = -6a+b+5c \\ z & = a-c \end{bmatrix},$$

so that $(x,y,z) = (10a-2b-7c,\ -6a+b+5c,\ a-c)$.

1.1.**19** a Note that the demand D_1 for product 1 increases with the increase of price P_2; likewise the demand D_2 for product 2 increases with the increase of price P_1. This indicates that the two products are competing; some people will switch if one of the products gets more expensive.

b Setting $D_1 = S_1$ and $D_2 = S_2$ we obtain the system $\begin{bmatrix} 70-2P_1+P_2 & = -14+3P_1 \\ 105+P_1-P_2 & = -7+2P_2 \end{bmatrix}$, or $\begin{bmatrix} -5P_1+P_2 & = -84 \\ P_1-3P_2 & = 112 \end{bmatrix}$, which yields the unique solution $P_1 = 26$ and $P_2 = 46$.

1.1.**20** The total demand for the product of Industry A is 1000 (the consumer demand) plus 0.1b (the demand from Industry B). The output a must meet this demand: $a = 1000 + 0.1b$.

Setting up a similar equation for Industry B we obtain the system $\begin{bmatrix} a & = 1000+0.1b \\ b & = 780+0.2a \end{bmatrix}$ or $\begin{bmatrix} a-0.1b & = 1000 \\ -0.2a+b & = 780 \end{bmatrix}$, which yields the unique solution $a = 1100$ and $b = 1000$.

1.1.**29** To assure that the graph goes through the point $(1, -1)$, we substitute $t = 1$ and $f(t) = -1$ into the equation $f(t) = a + bt + ct^2$ to give $-1 = a + b + c$.

Proceeding likewise for the two other points, we obtain the system $\begin{bmatrix} a+b+c & =-1 \\ a+2b+4c & =3 \\ a+3b+9c & =13 \end{bmatrix}$.

The solution is $a = 1$, $b = -5$, and $c = 3$, and the polynomial is $f(t) = 1 - 5t + 3t^2$. (See Figure 1.5.)

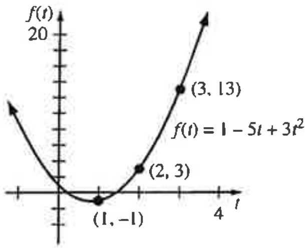

Figure 1.5: for Problem 1.1.29.

1.1.**30** Proceeding as in the previous exercise, we obtain the system $\begin{bmatrix} a+b+c & =p \\ a+2b+4c & =q \\ a+3b+9c & =r \end{bmatrix}$.

The unique solution is $\begin{bmatrix} a & =3p-3q+r \\ b & =-2.5p+4q-1.5r \\ c & =0.5p-q+0.5r \end{bmatrix}$.

Only one polynomial of degree 2 goes through the three given points, namely,

$f(t) = 3p - 3q + r + (-2.5p + 4q - 1.5r)t + (0.5p - q + 0.5r)t^2$.

1.1.**31** $f(t)$ is of the form $at^2 + bt + c$. So $f(1) = a(1^2) + b(1) + c = 3$, and $f(2) = a(2^2) + b(2) + c = 6$. Also, $f'(t) = 2at + b$, meaning that $f'(1) = 2a + b = 1$.

So we have a system of equations: $\begin{bmatrix} a+b+c=3 \\ 4a+2b+c=6 \\ 2a+b=1 \end{bmatrix}$

which reduces to $\begin{bmatrix} a=2 \\ b=-3 \\ c=4 \end{bmatrix}$.

1.1.47 Let x_1 = number of one-dollar bills, x_2 = the number of five-dollar bills, and x_3 = the number of ten-dollar bills. Then our system looks like: $\begin{bmatrix} x_1 + x_2 + x_3 & = 32 \\ x_1 + 5x_2 + 10x_3 & = 100 \end{bmatrix}$,

which reduces to give us solutions that fit: $x_1 = 15 + \frac{5}{4}x_3$, $x_2 = 17 - \frac{9}{4}x_3$, where x_3 can be chosen freely. Now let's keep in mind that x_1, x_2, and x_3 must be positive integers and see what conditions this imposes on the variable x_3. We see that since x_1 and x_2 must be integers, x_3 must be a multiple of 4. Furthermore, x_3 must be positive, and $x_2 = 17 - \frac{9}{4}x_3$ must be positive as well, meaning that $x_3 < \frac{68}{9}$. These constraints leave us with only one possibility, $x_3 = 4$, and we can compute the corresponding values $x_1 = 15 + \frac{5}{4}x_3 = 20$ and $x_2 = 17 - \frac{9}{4}x_3 = 8$.

Thus, we have 20 one-dollar bills, 8 five-dollar bills, and 4 ten-dollar bills.

* **1.2.18** a No, since the third column contains two leading ones.

b Yes

c No, since the third row contains a leading one, but the second row does not.

d Yes

1.2.22 Seven, namely $\begin{bmatrix}0&0&0\\0&0&0\end{bmatrix}$, $\begin{bmatrix}1&a&b\\0&0&0\end{bmatrix}$, $\begin{bmatrix}0&1&c\\0&0&0\end{bmatrix}$, $\begin{bmatrix}0&0&1\\0&0&0\end{bmatrix}$, $\begin{bmatrix}1&0&d\\0&1&e\end{bmatrix}$, $\begin{bmatrix}1&f&0\\0&0&1\end{bmatrix}$, $\begin{bmatrix}0&1&0\\0&0&1\end{bmatrix}$.

Here, $a, b, \ldots, f$ are arbitrary constants.

1.2.23 The conditions a, b, and c for the reduced row-echelon form correspond to the properties P1, P2, and P3 given on Page 13. The Gauss-Jordan algorithm, summarized on Page 15, guarantees that those properties are satisfied.

1.2.24 Yes; each elementary row operation is reversible, that is, it can be "undone." For example, the operation of row swapping can be undone by swapping the same rows again. The operation of dividing a row by a scalar can be reversed by multiplying the same row by the same scalar.

1.2.25 Yes; if A is transformed into B by a sequence of elementary row operations, then we can recover A from B by applying the inverse operations in the reversed order (compare with Exercise 24).

1.2.26 Yes, by Exercise 25, since rref(A) is obtained from A by a sequence of elementary row operations.

1.2.27 No; whatever elementary row operations you apply to $\begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix}$, you cannot make the last column equal to zero.

1.2.28 Suppose $(c_1, c_2, \ldots, c_n)$ is a solution of the system $\begin{bmatrix}a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1\\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2\\ \cdots\cdots\cdots\end{bmatrix}$.

To keep the notation simple, suppose we add k times the first equation to the second; then the second equation of the new system will be $(a_{21} + ka_{11})x_1 + \cdots + (a_{2n} + ka_{1n})x_n = b_2 + kb_1$.

We have to verify that $(c_1, c_2, \ldots, c_n)$ is a solution of this new equation. Indeed, $(a_{21} + ka_{11})c_1 + \cdots + (a_{2n} + ka_{1n})c_n = a_{21}c_1 + \cdots + a_{2n}c_n + k(a_{11}c_1 + \cdots + a_{1n}c_n) = b_2 + kb_1$.

We have shown that any solution of the "old" system is also a solution of the "new." To see that, conversely, any solution of the new system is also a solution of the old system, note that elementary row operations are reversible (compare with Exercise 24); we can obtain the old system by subtracting k times the first equation from the second equation of the new system.

1.2.29 Since the number of oxygen atoms remains constant, we must have $2a + b = 2c + 3d$.

Considering hydrogen and nitrogen as well, we obtain the system $\begin{bmatrix}2a + b = 2c + 3d\\ 2b = c + d\\ a = c + d\end{bmatrix}$ or

$$\begin{bmatrix}2a + b - 2c - 3d = 0\\ 2b - c - d = 0\\ a - c - d = 0\end{bmatrix}, \text{ which reduces to } \begin{bmatrix}a - 2d = 0\\ b - d = 0\\ c - d = 0\end{bmatrix}.$$

✱ 1.2.44 Kyle first must solve the following system: $\left[\begin{array}{cccc} x_1 & +x_2 & +x_3 & = 24 \\ 3x_1 & +2x_2 & +\frac{1}{2}x_3 & = 24 \end{array}\right]$.

This system reduces to $\left[\begin{array}{cccc} x_1 & & -1.5x_3 & = -24 \\ & x_2 & +2.5x_3 & = 48 \end{array}\right]$.

Thus, our solutions will be of the form $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1.5x_3 - 24 \\ -2.5x_3 + 48 \\ x_3 \end{bmatrix}$. Since all of our values must be non-negative integers (and x_3 must be even), we find the following solutions for $\begin{bmatrix} lilies \\ roses \\ daisies \end{bmatrix}$: $\begin{bmatrix} 0 \\ 8 \\ 16 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 3 \\ 18 \end{bmatrix}$. Since Olivia loves lilies, Kyle spends his 24 dollars on 3 lilies, 3 roses and 18 daisies.

★ **1.2.73** We let x_1 be the number of sheep, x_2 be the number of goats, and x_3 be the number of hogs. We can then use the two equations $\frac{1}{2}x_1+\frac{4}{3}x_2+\frac{7}{2}x_3 = 100$ and $x_1+x_2+x_3 = 100$ to generate the following augmented matrix:

$$\left[\begin{array}{ccc|c} \frac{1}{2} & \frac{4}{3} & \frac{7}{2} & 100 \\ 1 & 1 & 1 & 100 \end{array}\right]$$

then reduce it to $\left[\begin{array}{ccc|c} 1 & 0 & -\frac{13}{5} & 40 \\ 0 & 1 & \frac{18}{5} & 60 \end{array}\right]$.

With this, we see that our solutions will be of the form $\begin{bmatrix} 40+\frac{13}{5}s \\ 60-\frac{18}{5}s \\ s \end{bmatrix}$. Now all three components of this vector must be non-negative integers, meaning that s must be a non-negative multiple of 5 (that is, $s = 0, 5, 10, \ldots$) such that $60 - \frac{18}{5}s \geq 0$, or, $s \leq \frac{50}{3}$. This leaves the possible solutions $x_3 = s = 0, 5, 10$ and 15, and we can compute the corresponding values of $x_1 = 40 + \frac{13}{5}s$ and $x_2 = 60 - \frac{18}{5}s$ in each case.

So we find the following solutions: $\begin{bmatrix} 40 \\ 60 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 53 \\ 42 \\ 5 \end{bmatrix}$, $\begin{bmatrix} 66 \\ 24 \\ 10 \end{bmatrix}$ and $\begin{bmatrix} 79 \\ 6 \\ 15 \end{bmatrix}$.

Section 1.3

1.3.1 a No solution, since the last row indicates $0 = 1$.

b The unique solution is $x = 5,\ y = 6$.

c Infinitely many solutions; the first variable can be chosen freely.

⚹ **1.3.10** $\begin{bmatrix}1\\2\\3\end{bmatrix} \cdot \begin{bmatrix}1\\-2\\1\end{bmatrix} = 1 \cdot 1 + 2 \cdot (-2) + 3 \cdot 1 = 0$

1.3.27 By Theorem 1.3.4, rref $(A) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

1.3.28 There must be a leading one in each column: rref $(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

1.3.33 The ith component of $A\vec{x}$ is $[0\ 0\ \ldots\ 1\ \ldots\ 0]\begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_i \\ \cdots \\ x_n \end{bmatrix} = x_i$. (The 1 is in the ith position.)

Therefore, $A\vec{x} = \vec{x}$.

* **1.3.34** a $A\vec{e}_1 = \begin{bmatrix} a \\ d \\ g \end{bmatrix}$, $A\vec{e}_2 = \begin{bmatrix} b \\ e \\ h \end{bmatrix}$, and $A\vec{e}_3 = \begin{bmatrix} c \\ f \\ k \end{bmatrix}$.

b $B\vec{e}_1 = [\vec{v}_1\ \vec{v}_2\ \vec{v}_3]\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 1\vec{v}_1 + 0\vec{v}_2 + 0\vec{v}_3 = \vec{v}_1$.

Likewise, $B\vec{e}_2 = \vec{v}_2$ and $B\vec{e}_3 = \vec{v}_3$.

*1.3.47 a $\vec{x} = \vec{0}$ is a solution.

b This holds by part (a) and Theorem 1.3.3.

c If $\vec{x}_1$ and $\vec{x}_2$ are solutions, then $A\vec{x}_1 = \vec{0}$ and $A\vec{x}_2 = \vec{0}$.

Therefore, $A(\vec{x}_1 + \vec{x}_2) = A\vec{x}_1 + A\vec{x}_2 = \vec{0} + \vec{0} = \vec{0}$, so that $\vec{x}_1 + \vec{x}_2$ is a solution as well. Note that we have used Theorem 1.3.10a.

d $A(k\vec{x}) = k(A\vec{x}) = k\vec{0} = \vec{0}$

We have used Theorem 1.3.10b.