* 2.1.3 Not linear, since $y_2 = x_1x_3$ is nonlinear.

* 2.1.4 $A = \begin{bmatrix} 9 & 3 & -3 \\ 2 & -9 & 1 \\ 4 & -9 & -2 \\ 5 & 1 & 5 \end{bmatrix}$

2.1.5 By Theorem 2.1.2, the three columns of the 2×3 matrix A are $T(\vec{e}_1), T(\vec{e}_2)$, and $T(\vec{e}_3)$, so that

$$A = \begin{bmatrix} 7 & 6 & -13 \\ 11 & 9 & 17 \end{bmatrix}.$$

2.1.6 Note that $x_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, so that T is indeed linear, with matrix $\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$.

2.1.7 Note that $x_1\vec{v}_1 + \cdots + x_m\vec{v}_m = [\vec{v}_1 \dots \vec{v}_m] \begin{bmatrix} x_1 \\ \cdots \\ x_m \end{bmatrix}$, so that T is indeed linear, with matrix $[\vec{v}_1 \ \vec{v}_2 \ \cdots \ \vec{v}_m]$.

* 2.1.8 Reducing the system $\begin{bmatrix} x_1 + 7x_2 & = y_1 \\ 3x_1 + 20x_2 & = y_2 \end{bmatrix}$, we obtain $\begin{bmatrix} x_1 & & = & -20y_1 & + & 7y_2 \\ & x_2 & = & 3y_1 & - & y_2 \end{bmatrix}$.

2pts ✱ **2.1.13** a First suppose that $a \neq 0$. We have to attempt to solve the equation $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ for x_1 and x_2.

$$\begin{bmatrix} ax_1 & + & bx_2 & = & y_1 & \\ cx_1 & + & dx_2 & = & & y_2 \end{bmatrix} \div a \longrightarrow \begin{bmatrix} x_1 & + & \frac{b}{a}x_2 & = & \frac{1}{a}y_1 & \\ cx_1 & + & dx_2 & = & & y_2 \end{bmatrix} -c(I) \longrightarrow$$

$$\begin{bmatrix} x_1 & + & \frac{b}{a}x_2 & = & \frac{1}{a}y_1 & & \\ & & (d-\frac{bc}{a})x_2 & = & -\frac{c}{a}y_1 & + & y_2 \end{bmatrix} \longrightarrow$$

$$\begin{bmatrix} x_1 & + & \frac{b}{a}x_2 & = & \frac{1}{a}y_1 & & \\ & & (\frac{ad-bc}{a})x_2 & = & -\frac{c}{a}y_1 & + & y_2 \end{bmatrix}$$

We can solve this system for x_1 and x_2 if (and only if) $ad - bc \neq 0$, as claimed.

If $a = 0$, then we have to consider the system

$$\begin{bmatrix} & & bx_2 & = & y_1 & \\ cx_1 & + & dx_2 & = & & y_2 \end{bmatrix} \text{swap} : I \leftrightarrow II \begin{bmatrix} cx_1 & + & dx_2 & = & & y_2 \\ & & bx_2 & = & y_1 & \end{bmatrix}$$

We can solve for x_1 and x_2 provided that both b and c are nonzero, that is if $bc \neq 0$. Since $a = 0$, this means that $ad - bc \neq 0$, as claimed.

b First suppose that $ad - bc \neq 0$ and $a \neq 0$. Let $D = ad - bc$ for simplicity. We continue our work in part (a):

$$\begin{bmatrix} x_1 & + & \frac{b}{a}x_2 & = & \frac{1}{a}y_1 & & \\ & & \frac{D}{a}x_2 & = & -\frac{c}{a}y_1 & + & y_2 \end{bmatrix} \cdot \frac{a}{D} \longrightarrow$$

$$\begin{bmatrix} x_1 & + & \frac{b}{a}x_2 & = & \frac{1}{a}y_1 & & \\ & & x_2 & = & -\frac{c}{D}y_1 & + & \frac{a}{D}y_2 \end{bmatrix} -\frac{b}{a}(II) \longrightarrow$$

$$\begin{bmatrix} x_1 & & = & (\frac{1}{a} + \frac{bc}{aD})y_1 & - & \frac{b}{D}y_2 \\ & x_2 & = & -\frac{c}{D}y_1 & + & \frac{a}{D}y_2 \end{bmatrix}$$

$$\begin{bmatrix} x_1 & & = & \frac{d}{D}y_1 & - & \frac{b}{D}y_2 \\ & x_2 & = & -\frac{c}{D}y_1 & + & \frac{a}{D}y_2 \end{bmatrix}$$

(Note that $\frac{1}{a} + \frac{bc}{aD} = \frac{D+bc}{aD} = \frac{ad}{aD} = \frac{d}{D}$.)

It follows that $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, as claimed. If $ad - bc \neq 0$ and $a = 0$, then we have to solve the system

$$\left[\begin{array}{llr} cx_1+ & dx_2 & = y_2 \\ & bx_2 & = y_1 \end{array}\right] \begin{array}{l} \div c \\ \div b \end{array}$$

$$\left[\begin{array}{llr} x_1+ & \frac{d}{c}x_2 & = \frac{1}{c}y_2 \\ & x_2 & = \frac{1}{b}y_1 \end{array}\right] -\tfrac{d}{c}(II)$$

$$\left[\begin{array}{llcll} x_1 & & = & -\frac{d}{bc}y_1 & +\frac{1}{c}y_2 \\ & x_2 & = & \frac{1}{b}y_1 & \end{array}\right]$$

It follows that $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \begin{bmatrix} -\frac{d}{bc} & \frac{1}{c} \\ \frac{1}{b} & 0 \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ (recall that $a = 0$), as claimed.

2.1.25 The matrix represents a scaling by the factor of 2. (See Figure 2.10.)

2.1.26 This matrix represents a reflection about the line $x_2 = x_1$. (See Figure 2.11.)

2.1.27 This matrix represents a reflection about the $\vec{e}_1$ axis. (See Figure 2.12.)

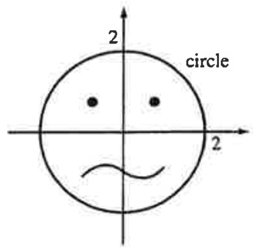

Figure 2.10: for Problem 2.1.25.

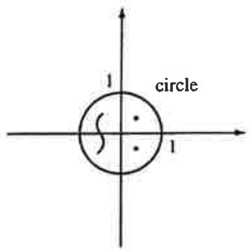

Figure 2.11: for Problem 2.1.26.

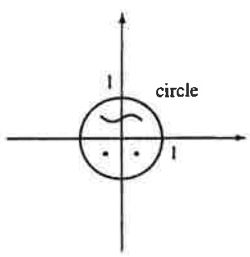

Figure 2.12: for Problem 2.1.27.

2.1.**28** If $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, then $A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 2x_2 \end{bmatrix}$, so that the x_2 component is multiplied by 2, while the x_1 component remains unchanged. (See Figure 2.13.)

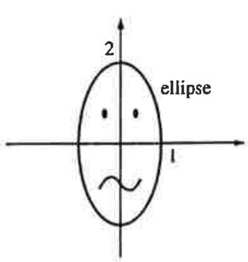

Figure 2.13: for Problem 2.1.28.

✗ 2.1.**32** Using Theorem 2.1.2, we find $A = \begin{bmatrix} 3 & 0 & \cdot & 0 \\ 0 & 3 & \cdot & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 3 \end{bmatrix}$. This matrix has 3's on the diagonal and 0's everywhere else.

★ 2.1.**37** Since $\vec{x} = \vec{v} + k(\vec{w} - \vec{v})$, we have $T(\vec{x}) = T(\vec{v} + k(\vec{w} - \vec{v})) = T(\vec{v}) + k(T(\vec{w}) - T(\vec{v}))$, by Theorem 2.1.3

Since k is between 0 and 1, the tip of this vector $T(\vec{x})$ is on the line segment connecting the tips of $T(\vec{v})$ and $T(\vec{w})$. (See Figure 2.18.)

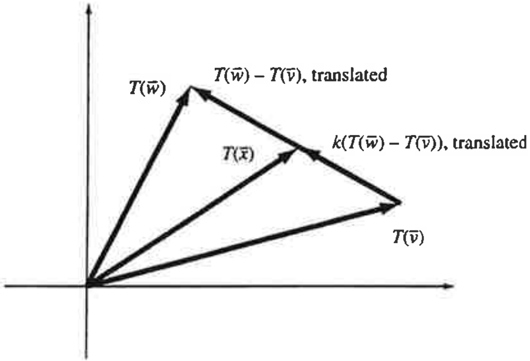

Figure 2.18: for Problem 2.1.37.

* 2.1.48 Let $\vec{x}$ be some vector in $\mathbb{R}^2$. Since $\vec{v}_1$ and $\vec{v}_2$ are not parallel, we can write $\vec{x}$ in terms of components of $\vec{v}_1$ and $\vec{v}_2$. So, let c_1 and c_2 be scalars such that $\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2$. Then, by Theorem 2.1.3, $T(\vec{x}) = T(c_1\vec{v}_1 + c_2\vec{v}_2) = T(c_1\vec{v}_1) + T(c_2\vec{v}_2) = c_1T(\vec{v}_1) + c_2T(\vec{v}_2) = c_1L(\vec{v}_1) + c_2L(\vec{v}_2) = L(c_1\vec{v}_1 + c_2\vec{v}_2) = L(\vec{x})$. So $T(\vec{x}) = L(\vec{x})$ for all $\vec{x}$ in $\mathbb{R}^2$.

* **2.1.50** a Let $\begin{bmatrix} p \\ s \end{bmatrix} = \begin{bmatrix} \text{mass of the platinum alloy} \\ \text{mass of the silver alloy} \end{bmatrix}$. Using the definition density = mass/volume, or volume = mass/density, we can set up the system:

$\begin{bmatrix} p & +s & = & 5,000 \\ \frac{p}{20} & +\frac{s}{10} & = & 370 \end{bmatrix}$, with the solution $p = 2,600$ and $s = 2,400$. We see that the platinum alloy makes up only 52 percent of the crown; this gold smith is a crook!

b We seek the matrix A such that $A \begin{bmatrix} p \\ s \end{bmatrix} = \begin{bmatrix} \text{total mass} \\ \text{total volume} \end{bmatrix} = \begin{bmatrix} p+s \\ \frac{p}{20} + \frac{s}{10} \end{bmatrix}$. Thus $A = \begin{bmatrix} 1 & 1 \\ \frac{1}{20} & \frac{1}{10} \end{bmatrix}$.

c Yes. By Exercise 13, $A^{-1} = \begin{bmatrix} 2 & -20 \\ -1 & 20 \end{bmatrix}$. Applied to the case considered in part a, we find that $\begin{bmatrix} p \\ s \end{bmatrix} = A^{-1} \begin{bmatrix} \text{total mass} \\ \text{total volume} \end{bmatrix} = \begin{bmatrix} 2 & -20 \\ -1 & 20 \end{bmatrix} \begin{bmatrix} 5,000 \\ 370 \end{bmatrix} = \begin{bmatrix} 2,600 \\ 2,400 \end{bmatrix}$, confirming our answer in part a.

* 2.3.4 $\begin{bmatrix} 2 & 2 \\ 2 & 0 \\ 7 & 4 \end{bmatrix}$

✱ **2.3.14** $A^2 = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$, $BC = [14 \;\; 8 \;\; 2]$, $BD = [6]$, $C^2 = \begin{bmatrix} -2 & -2 & -2 \\ 4 & 1 & -2 \\ 10 & 4 & -2 \end{bmatrix}$, $CD = \begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix}$, $DB = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$,

$DE = \begin{bmatrix} 5 \\ 5 \\ 5 \end{bmatrix}$, $EB = [5 \;\; 10 \;\; 15]$, $E^2 = [25]$

2.3.15 $$\left[\begin{array}{c|c} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}[3] & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}[4] \\ \hline [1 \;\; 3]\begin{bmatrix} 1 \\ 2 \end{bmatrix} + [4][3] & [1 \;\; 3]\begin{bmatrix} 0 \\ 0 \end{bmatrix} + [4][4] \end{array}\right] = \left[\begin{array}{c|c} \begin{bmatrix} 1 \\ 2 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \hline [19] & [16] \end{array}\right] = \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 19 & 16 \end{bmatrix}$$

2.3.16 $$\left[\begin{array}{c|c} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \\ \hline \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \end{array}\right] = \begin{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} & \begin{bmatrix} 3 & 5 \\ 7 & 9 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 5 \\ 3 & 4 & 7 & 9 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 3 & 4 \end{bmatrix}$$

2.3.17 We must find all S such that $SA = AS$, or $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

So $\begin{bmatrix} a & 2b \\ c & 2d \end{bmatrix} = \begin{bmatrix} a & b \\ 2c & 2d \end{bmatrix}$, meaning that $b = 2b$ and $c = 2c$, so b and c must be zero.

We see that all diagonal matrices (those of the form $\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$) commute with $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$.

✱ **2.3.18** As in Exercise 2.3.17, we let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Now we want $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

So, $\begin{bmatrix} a & 2a+b \\ c & 2c+d \end{bmatrix} = \begin{bmatrix} a+2c & b+2d \\ c & d \end{bmatrix}$, revealing that $c = 0$ (since $a + 2c = a$) and $a = d$ (since $b + 2d = 2a + b$).

Thus B is any matrix of the form $\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$.

2.3.32 Let $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then we want $X \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} X$, or $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, or $\begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$, meaning that $b = c = 0$. Also, we want $X \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} X$, or $\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$, or $\begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & d \\ 0 & 0 \end{bmatrix}$ so $a = d$. Thus, $X = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} = aI_2$ must be a multiple of the identity matrix. (X will then commute with any 2×2 matrix M, since $XM = aM = MX$.)

2.3.33 $A^2 = I_2$, $A^3 = A, A^4 = I_2$. The power A^n alternates between $A = -I_2$ and I_2. The matrix A describes a reflection about the origin. Alternatively one can say A represents a rotation by $180° = \pi$. Since A^2 is the identity, A^{1000} is the identity and $A^{1001} = A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$.

2.3.34 $A^2 = I_2$, $A^3 = A, A^4 = I_2$. The power A^n alternates between A and I_2. The matrix A describes a reflection about the x axis. Because A^2 is the identity, A^{1000} is the identity and $A^{1001} = A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

2.3.35 $A^2 = I_2$, $A^3 = A, A^4 = I_2$. The power A^n alternates between A and I_2. The matrix A describes a reflection about the diagonal $x = y$. Because A^2 is the identity, A^{1000} is the identity and $A^{1001} = A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

2.3.36 $A^2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, $A^3 = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$ and $A^4 = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$. The power A^n represents a horizontal shear along the x-axis. The shear strength increases linearly in n. We have $A^{1001} = \begin{bmatrix} 1 & 1001 \\ 0 & 1 \end{bmatrix}$.