* 2.2.4 By Theorem 2.2.4, this is a rotation combined with a scaling. The transformation rotates 45 degrees counter-clockwise, and has a scaling factor of $\sqrt{2}$.

2.2.10 By Theorem 2.2.1, $\text{proj}_L\vec{x} = (\vec{u}\cdot\vec{x})\vec{u}$, where $\vec{u}$ is a unit vector on L. We can choose $\vec{u} = \frac{1}{5}\begin{bmatrix}4\\3\end{bmatrix} = \begin{bmatrix}0.8\\0.6\end{bmatrix}$. Then

$$\text{proj}_L\begin{bmatrix}x_1\\x_2\end{bmatrix} = \left(\begin{bmatrix}0.8\\0.6\end{bmatrix}\cdot\begin{bmatrix}x_1\\x_2\end{bmatrix}\right)\begin{bmatrix}0.8\\0.6\end{bmatrix} = (0.8x_1+0.6x_2)\begin{bmatrix}0.8\\0.6\end{bmatrix} = \begin{bmatrix}0.64x_1+0.48x_2\\0.48x_1+0.36x_2\end{bmatrix} = \begin{bmatrix}0.64 & 0.48\\0.48 & 0.36\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}.$$

The matrix is $A = \begin{bmatrix}0.64 & 0.48\\0.48 & 0.36\end{bmatrix}$.

2.2.11 In Exercise 10 we found the matrix $A = \begin{bmatrix}0.64 & 0.48\\0.48 & 0.36\end{bmatrix}$ of the projection onto the line L. By Theorem 2.2.2, $\text{ref}_L\vec{x} = 2(\text{proj}_L\vec{x}) - \vec{x} = 2A\vec{x} - \vec{x} = (2A - I_2)\vec{x}$, so that the matrix of the reflection is $2A - I_2 = \begin{bmatrix}0.28 & 0.96\\0.96 & -0.28\end{bmatrix}$.

2.2.12 Let $\vec{u} = (1/||\vec{w}||)\vec{w}$ be the unit vector in the direction of $\vec{w}$. It has the components $u_1 = w_1/\sqrt{w_1^2+w_2^2}$ and $u_2 = \frac{w_2}{\sqrt{w_1^2+w_2^2}}$. On Pages 57/58, we see that the matrix representing the projection is

$$\begin{bmatrix}u_1^2 & u_1u_2\\u_1u_2 & u_2^2\end{bmatrix}.$$

This can be written as

$$\frac{1}{w_1^2+w_2^2}\begin{bmatrix}w_1^2 & w_1w_2\\w_1w_2 & w_2^2\end{bmatrix},$$

as claimed.

2.2.13 By Theorem 2.2.2,

$$\text{ref}_L\begin{bmatrix}x_1\\x_2\end{bmatrix} = 2\left(\begin{bmatrix}u_1\\u_2\end{bmatrix}\cdot\begin{bmatrix}x_1\\x_2\end{bmatrix}\right)\begin{bmatrix}u_1\\u_2\end{bmatrix} - \begin{bmatrix}x_1\\x_2\end{bmatrix}$$
$$= 2(u_1x_1+u_2x_2)\begin{bmatrix}u_1\\u_2\end{bmatrix} - \begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}(2u_1^2-1)x_1+2u_1u_2x_2\\2u_1u_2x_1+(2u_2^2-1)x_2\end{bmatrix}.$$

The matrix is $A = \begin{bmatrix}a & b\\c & d\end{bmatrix} = \begin{bmatrix}2u_1^2-1 & 2u_1u_2\\2u_1u_2 & 2u_2^2-1\end{bmatrix}$. Note that the sum of the diagonal entries is $a + d = 2(u_1^2+u_2^2) - 2 = 0$, since $\vec{u}$ is a unit vector. It follows that $d = -a$. Since $c = b$, A is of the form $\begin{bmatrix}a & b\\b & -a\end{bmatrix}$. Also, $a^2+b^2 = (2u_1^2-1)^2 + 4u_1^2u_2^2 = 4u_1^4 - 4u_1^2 + 1 + 4u_1^2(1-u_1^2) = 1$, as claimed.

2.2.14 a Proceeding as on Page 57/58 in the text, we find that A is the matrix whose ijth entry is u_iu_j:

$$A = \begin{bmatrix}u_1^2 & u_1u_2 & u_1u_3\\u_2u_1 & u_2^2 & u_2u_3\\u_nu_1 & u_nu_2 & u_3^2\end{bmatrix}$$

b The sum of the diagonal entries is $u_1^2+u_2^2+u_3^2 = 1$, since $\vec{u}$ is a unit vector.

2.2.17 We want, $\begin{bmatrix} a & b \\ b & -a \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} av_1 & +bv_2 \\ bv_1 & -av_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$.

Now, $(a-1)v_1 + bv_2 = 0$ and $bv_1 - (a+1)v_2$, which is a system with solutions of the form $\begin{bmatrix} bt \\ (1-a)t \end{bmatrix}$, where t is an arbitrary constant.

Let's choose $t = 1$, making $\vec{v} = \begin{bmatrix} b \\ 1-a \end{bmatrix}$.

Similarly, we want $A\vec{w} = -\vec{w}$. We perform a computation as above to reveal $\vec{w} = \begin{bmatrix} a-1 \\ b \end{bmatrix}$ as a possible choice. A quick check of $\vec{v} \cdot \vec{w} = 0$ reveals that they are indeed perpendicular.

Now, any vector $\vec{x}$ in $\mathbb{R}$ can be written in terms of components with respect to $L = \text{span}(\vec{v})$ as $\vec{x} = \vec{x}^{||} + \vec{x}^{\perp} = c\vec{v} + d\vec{w}$. Then, $T(\vec{x}) = A\vec{x} = A(c\vec{v} + d\vec{w}) = A(c\vec{v}) + A(d\vec{w}) = cA\vec{v} + dA\vec{w} = c\vec{v} - d\vec{w} = \vec{x}^{||} - \vec{x}^{\perp} = \text{ref}_L(\vec{x})$, by Definition 2.2.2.

(The vectors $\vec{v}$ and $\vec{w}$ constructed above are both zero in the special case that $a = 1$ and $b = 0$. In that case, we can let $\vec{v} = \vec{e}_1$ and $\vec{w} = \vec{e}_2$ instead.)

2.2.18 From Exercise 17, we know that the reflection is about the line parallel to $\vec{v} = \begin{bmatrix} b \\ 1-a \end{bmatrix} = \begin{bmatrix} 0.8 \\ 0.4 \end{bmatrix} = 0.4 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

So, every point on this line can be described as $\begin{bmatrix} x \\ y \end{bmatrix} = k \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. So, $y = k = \frac{1}{2}x$, and $y = \frac{1}{2}x$ is the line we are looking for.

2.2.19 $T(\vec{e}_1) = \vec{e}_1$, $T(\vec{e}_2) = \vec{e}_2$, and $T(\vec{e}_3) = \vec{0}$, so that the matrix is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

2.2.20 $T(\vec{e}_1) = \vec{e}_1$, $T(\vec{e}_2) = -\vec{e}_2$, and $T(\vec{e}_3) = \vec{e}_3$, so that the matrix is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

2pts ★ **2.2.26** a $\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 2k \\ -k \end{bmatrix} = \begin{bmatrix} 8 \\ -4 \end{bmatrix}$. So $k = 4$ and $A = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$.

b This is the orthogonal projection onto the horizontal axis, with matrix $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

c $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}\begin{bmatrix} 0 \\ 5 \end{bmatrix} = \begin{bmatrix} -5b \\ 5a \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$. So $a = \frac{4}{5}, b = -\frac{3}{5}$, and $C = \begin{bmatrix} \frac{4}{5} & \frac{3}{5} \\ -\frac{3}{5} & \frac{4}{5} \end{bmatrix}$. Note that $a^2 + b^2 = 1$, as required for a rotation matrix.

d Since the x_1 term is being modified, this must be a horizontal shear.

Then $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1+3k \\ 3 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$. So $k = 2$ and $D = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$.

e $\begin{bmatrix} a & b \\ b & -a \end{bmatrix}\begin{bmatrix} 7 \\ 1 \end{bmatrix} = \begin{bmatrix} 7a+b \\ 7b-a \end{bmatrix} = \begin{bmatrix} -5 \\ 5 \end{bmatrix}$. So $a = -\frac{4}{5}, b = \frac{3}{5}$, and $E = \begin{bmatrix} -\frac{4}{5} & \frac{3}{5} \\ \frac{3}{5} & \frac{4}{5} \end{bmatrix}$. Note that $a^2 + b^2 = 1$, as required for a reflection matrix.

2.2.32 a See Figure 2.29.

b Compute $D\vec{v} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\beta \\ \sin\beta \end{bmatrix} = \begin{bmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta \\ \sin\alpha\cos\beta + \cos\alpha\sin\beta \end{bmatrix}$.

Comparing this result with our finding in part (a), we get the addition theorems

$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$\sin(\alpha+\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

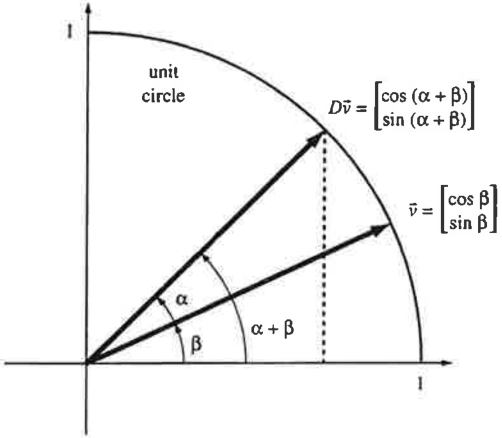

Figure 2.29: for Problem 2.2.32a.

★ 2.2.44 By Exercise 1.1.13b, $A^{-1} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}^{-1} = \frac{1}{a^2+b^2}\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$.

If A represents a rotation through θ followed by a scaling by r, then A^{-1} represents a rotation through $-\theta$ followed by a scaling by $\frac{1}{r}$. (See Figure 2.36.)

* 2.4.4 Use Theorem 2.4.5; the inverse is $\begin{bmatrix} \frac{3}{2} & -1 & \frac{1}{2} \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ -\frac{3}{2} & 1 & \frac{1}{2} \end{bmatrix}$.

2.4.20 Solving for x_1, x_2, and x_3 in terms of y_1, y_2, and y_3 we find that

$$\begin{aligned} x_1 &= -8y_1 - 15y_2 + 12y_3 \\ x_2 &= 4y_1 + 6y_2 - 5y_3 \\ x_3 &= -y_1 - y_2 + y_3 \end{aligned}$$

✱ **2.4.30** Use Theorem 2.4.3:

$$\begin{bmatrix} 0 & 1 & b \\ -1 & 0 & c \\ -b & -c & 0 \end{bmatrix} \xrightarrow{I \leftrightarrow II} \begin{bmatrix} -1 & 0 & c \\ 0 & 1 & b \\ -b & -c & 0 \end{bmatrix} \xrightarrow{\div(-1)} \begin{bmatrix} 1 & 0 & -c \\ 0 & 1 & b \\ -b & -c & 0 \end{bmatrix} \begin{matrix} \\ \\ +b(I)+c(II) \end{matrix} \longrightarrow \begin{bmatrix} 1 & 0 & -c \\ 0 & 1 & b \\ 0 & 0 & 0 \end{bmatrix}$$

This matrix fails to be invertible, regarless of the values of b and c.

* 2.4.**34** a By Theorem 2.4.3, A is invertible if (and only if) $a, b,$ and c are all nonzero. In this case, $A^{-1} = \begin{bmatrix} \frac{1}{a} & 0 & 0 \\ 0 & \frac{1}{b} & 0 \\ 0 & 0 & \frac{1}{c} \end{bmatrix}$.

b In general, a diagonal matrix is invertible if (and only if) all of its diagonal entries are nonzero.

2.4.**35** a A is invertible if (and only if) all its diagonal entries, $a, d,$ and f, are nonzero.

b As in part (a): if all the diagonal entries are nonzero.

c Yes, A^{-1} will be upper triangular as well; as you construct rref$[A \vdots I_n]$, you will perform only the following row operations:

- divide rows by scalars
- subtract a multiple of the jth row from the ith row, where $j > i$.

Applying these operations to I_n, you end up with an upper triangular matrix.

d As in part (b): if all diagonal entries are nonzero.

* 2.4.**36** If a matrix A can be transformed into B by elementary row operations, then A is invertible if (and only if) B is invertible. The claim now follows from Exercise 35, where we show that a triangular matrix is invertible if (and only if) its diagonal entries are nonzero.

2pts * 2.4.44 a $\text{rref}(M_4) = \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, so that $\text{rank}(M_4) = 2$.

b To simplify the notation, we introduce the row vectors $\vec{v} = [1\ 1\ \ldots\ 1]$ and $\vec{w} = [0\ n\ 2n\ \ldots\ (n-1)n]$ with n components.

Then we can write M_n in terms of its rows as $M_n = \begin{bmatrix} \vec{v}+\vec{w} \\ 2\vec{v}+\vec{w} \\ \cdots \\ n\vec{v}+\vec{w} \end{bmatrix} \begin{matrix} \\ -2(I) \\ \cdots \\ -n(I) \end{matrix}$.

Applying the Gauss-Jordan algorithm to the first column we get $\begin{bmatrix} \vec{v}+\vec{w} \\ -\vec{w} \\ -2\vec{w} \\ \cdots \\ -(n-1)\vec{w} \end{bmatrix}$.

All the rows below the second are scalar multiples of the second; therefore, $\text{rank}(M_n) = 2$.

c By part (b), the matrix M_n is invertible only if $n = 1$ or $n = 2$.

2.4.77 We want A such that $A\vec{v}_i = \vec{w}_i$, for $i = 1, 2, \ldots, m$, or $A[\vec{v}_1\ \vec{v}_2\ \ldots\ \vec{v}_m] = [\vec{w}_1\ \vec{w}_2\ \ldots\ \vec{w}_m]$, or $AS = B$.

Multiplying by S^{-1} from the right we find the unique solution $A = BS^{-1}$.

* **2.4.78** Use the result of Exercise 2.4.77, with $S = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 7 & 1 \\ 5 & 2 \\ 3 & 3 \end{bmatrix}$;

$$A = BS^{-1} = \begin{bmatrix} 33 & -13 \\ 21 & -8 \\ 9 & -3 \end{bmatrix}$$