3.1.4 Find all $\vec{x}$ such that $A\vec{x} = \vec{0}$, or $x_1 + 2x_2 + 3x_2 = 0$.

The solutions are of the form $\begin{bmatrix} x_1 \\ x_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2t - 3r \\ t \\ r \end{bmatrix} = t\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + r\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$, so that

$$\ker(A) = \operatorname{span}\left(\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right).$$

3.1.10 Solving the system $A\vec{x} = \vec{0}$ we find that $\ker(A) = \text{span}\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}$.

3.1.11 Solving the system $A\vec{x} = \vec{0}$ we find that $\ker(A) = \text{span}\begin{bmatrix} -2 \\ 3 \\ 1 \\ 0 \end{bmatrix}$.

3.1.12 Solving the system $A\vec{x} = \vec{0}$ we find that $\ker(A) = \text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}\right)$.

3.1.13 Solving the system $A\vec{x} = \vec{0}$ we find that $\ker(A) = \text{span}\left(\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ -2 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right)$.

3.1.14 By Theorem 3.1.3, the image of A is the span of the column vectors of A:

$$\text{im}(A) = \text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}\right).$$

3.1.22 Compare with the solution to Exercise 21.

$$\left[\begin{array}{cc:c} 2 & 1 & 3 \\ 3 & 4 & 2 \\ 6 & 5 & 7 \end{array}\right] \longrightarrow \left[\begin{array}{cc:c} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right]$$

3.1.30 By Theorem 3.1.3, $A = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$ does the job. There are many other possible answers: any nonzero $2 \times n$ matrix A whose column vectors are scalar multiples of vector $\begin{bmatrix} 1 \\ 5 \end{bmatrix}$.

3.1.**38** a If a vector $\vec{x}$ is in $\ker(A^k)$, that is, $A^k\vec{x}=\vec{0}$, then $\vec{x}$ is also in $\ker(A^{k+1})$, since $A^{k+1}\vec{x}=AA^k\vec{x}=A\vec{0}=\vec{0}$.

Therefore, $\ker(A)\subseteq\ker(A^2)\subseteq\ker(A^3)\subseteq\ldots$

Exercise 37 shows that these kernels need not be equal.

b If a vector $\vec{y}$ is in $\mathrm{im}(A^{k+1})$, that is, $\vec{y}=A^{k+1}\vec{x}$ for some $\vec{x}$, then $\vec{y}$ is also in $\mathrm{im}(A^k)$, since we can write $\vec{y}=A^k(A\vec{x})$. Therefore, $\mathrm{im}(A)\supseteq\mathrm{im}(A^2)\supseteq\mathrm{im}(A^3)\supseteq\ldots$.

Exercise 37 shows that these images need not be equal.

3.1.**39** a If a vector $\vec{x}$ is in $\ker(B)$, that is, $B\vec{x}=\vec{0}$, then $\vec{x}$ is also in $\ker(AB)$, since $AB(\vec{x})=A(B\vec{x})=A\vec{0}=\vec{0}$:

$\ker(B)\subseteq\ker(AB)$.

Exercise 37 (with $A=B$) illustrates that these kernels need not be equal.

b If a vector $\vec{y}$ is in $\mathrm{im}(AB)$, that is, $\vec{y}=AB\vec{x}$ for some $\vec{x}$, then $\vec{y}$ is also in $\mathrm{im}(A)$, since we can write

$\vec{y}=A(B\vec{x})$:

$\mathrm{im}(AB)\subseteq\mathrm{im}(A)$.

Exercise 37 (with $A=B$) illustrates that these images need not be equal.

3.1.**40** For any $\vec{x}$ in $\mathbb{R}^m$, the vector $B\vec{x}$ is in $\mathrm{im}(B)=\ker(A)$, so that $AB\vec{x}=\vec{0}$. If we apply this fact to $\vec{x}=\vec{e}_1,\vec{e}_2,\ldots,\vec{e}_m$, we find that all the columns of the matrix AB are zero, so that $AB=0$.

3.1.44 a Yes; by construction of the echelon form, the systems $A\vec{x} = \vec{0}$ and $B\vec{x} = \vec{0}$ have the same solutions (it is the whole point of Gaussian elimination not to change the solutions of a system).

b No; as a counterexample, consider $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, with $\text{im}(A) = \text{span}(\vec{e}_2)$, but $B = \text{rref}(A) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, with $\text{im}(B) = \text{span}(\vec{e}_1)$.

3.2.2 Not a subspace, since W contains the vector $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ but not the vector $(-1)\vec{v} = \begin{bmatrix} -1 \\ -2 \\ -3 \end{bmatrix}$.

3.2.3 $W = \text{im} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ is a subspace of $\mathbb{R}^3$, by Theorem 3.2.2.

3.2.4 $\text{span}(\vec{v}_1, \ldots, \vec{v}_m) = \text{im}[\vec{v}_1 \ \ldots \ \vec{v}_m]$ is a subspace of $\mathbb{R}^n$, by Theorem 3.2.2.

3.2.5 We have subspaces $\{\vec{0}\}$, $\mathbb{R}^3$, and all lines and planes (through the origin). To prove this, mimic the reasoning in Example 2.

3.2.6 a Yes!

- The zero vector is in $V \cap W$, since $\vec{0}$ is in both V and W.
- If $\vec{x}$ and $\vec{y}$ are in $V \cap W$, then both $\vec{x}$ and $\vec{y}$ are in V, so that $\vec{x} + \vec{y}$ is in V as well, since V is a subspace of $\mathbb{R}^n$. Likewise, $\vec{x} + \vec{y}$ is in W, so that $\vec{x} + \vec{y}$ is in $V \cap W$.
- If $\vec{x}$ is in $V \cap W$ and k is an arbitrary scalar, then $k\vec{x}$ is in both V and W, since they are subspaces of $\mathbb{R}^n$. Therefore, $k\vec{x}$ is in $V \cap W$.

b No; as a counterexample consider $V = \text{span}(\vec{e}_1)$ and $W = \text{span}(\vec{e}_2)$ in $\mathbb{R}^2$.

3.2.18 Linearly dependent, since rref$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 7 \\ 1 & 4 & 10 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. So, we find that the vector $\begin{bmatrix} 1 \\ 4 \\ 7 \\ 10 \end{bmatrix}$ turns out to be redundant.

3.2.34 The fact that $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ is in ker(A) means that

$$A \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = [\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3 \ \vec{v}_4] \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \vec{v}_1 + 2\vec{v}_2 + 3\vec{v}_3 + 4\vec{v}_4 = \vec{0}, \text{ so that } \vec{v}_4 = -\tfrac{1}{4}\vec{v}_1 - \tfrac{1}{2}\vec{v}_2 - \tfrac{3}{4}\vec{v}_3.$$

3.2.35 If $\vec{v}_i$ is a linear combination of the other vectors in the list, $\vec{v}_i = c_1\vec{v}_1 + \cdots + c_{i-1}\vec{v}_{i-1} + c_{i+1}\vec{v}_{i+1} + \cdots + c_n\vec{v}_n$, then we can subtract $\vec{v}_i$ from both sides to generate a nontrivial relation (the coefficient of $\vec{v}_i$ will be -1).

Conversely, if there is a nontrivial relation $c_1\vec{v}_1 + \cdots + c_i\vec{v}_i + \cdots + c_n\vec{v}_n = \vec{0}$, with $c_i \neq 0$, then we can solve for vector $\vec{v}_i$ and thus express $\vec{v}_i$ as a linear combination of the other vectors in the list.

3.2.36 Yes; we know that there is a nontrivial relation $c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_m\vec{v}_m = \vec{0}$.

Now apply the transformation T to the vectors on both sides, and use linearity:

$T(c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_m\vec{v}_m) = T(\vec{0})$, so that $c_1T(\vec{v}_1) + c_2T(\vec{v}_2) + \cdots + c_mT(\vec{v}_m) = \vec{0}$.

3.2.40 Yes; by Theorem 3.2.8, $\ker(A) = \{\vec{0}\}$ and $\ker(B) = \{\vec{0}\}$. Then $\ker(AB) = \{\vec{0}\}$ by Exercise 3.1.51, so that the columns of AB are linearly independent, by Theorem 3.2.8.

3.2.41 To show that the columns of B are linearly independent, we show that $\ker(B) = \{\vec{0}\}$. Indeed, if $B\vec{x} = \vec{0}$, then $AB\vec{x} = A\vec{0} = \vec{0}$, so that $\vec{x} = \vec{0}$ (since $AB = I_m$).

By Theorem 3.2.8, $\text{rank}(B) = \#\text{ columns} = m$, so that $m \leq n$ and in fact $m < n$ (we are told that $m \neq n$). This implies that the rank of the $m \times n$ matrix A is less than n, so that the columns of A are linearly dependent (by Theorem 3.2.8).

3.2.42 We can use the hint and form the dot product of $\vec{v}_i$ and both sides of the relation

$c_1\vec{v}_1 + \cdots + c_i\vec{v}_i + \cdots + c_m\vec{v}_m = \vec{0}$:

$(c_1\vec{v}_1 + \cdots + c_i\vec{v}_i + \cdots + c_m\vec{v}_m) \cdot \vec{v}_i = \vec{0} \cdot \vec{v}_i$, so that $c_1(\vec{v}_1 \cdot \vec{v}_i) + \cdots + c_i(\vec{v}_i \cdot \vec{v}_i) + \cdots + c_m(\vec{v}_m \cdot \vec{v}_i) = 0$.

Since $\vec{v}_i$ is perpendicular to all the other $\vec{v}_j$, we will have $\vec{v}_i \cdot \vec{v}_j = 0$ whenever $j \neq i$; since $\vec{v}_i$ is a unit vector, we will have $\vec{v}_i \cdot \vec{v}_i = 1$. Therefore, the equation above simplifies to $c_i = 0$.

Since this reasoning applies to all $i = 1, \ldots, m$, we have only the trivial relation among the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$, so that these vectors are linearly independent, as claimed.

3.2.43 Consider a linear relation $c_1\vec{v}_1 + c_2(\vec{v}_1 + \vec{v}_2) + c_3(\vec{v}_1 + \vec{v}_2 + \vec{v}_3) = \vec{0}$, or, $(c_1 + c_2 + c_3)\vec{v}_1 + (c_2 + c_3)\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$. Since there is only the trivial relation among the vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$, we must have $c_1 + c_2 + c_3 = c_2 + c_3 = c_3 = 0$, so that $c_3 = 0$ and then $c_2 = 0$ and then $c_1 = 0$, as claimed.

3.2.44 Yes; this is a special case of Exercise 40 (recall that $\ker(A) = \{\vec{0}\}$, by Theorem 3.1.7b).

3.2.50 The verification of the three properties listed in Definition 3.2.1 is straightforward. Alternatively, we can choose a basis $\vec{v}_1, \ldots, \vec{v}_p$ of V and a basis $\vec{w}_1, \ldots, \vec{w}_q$ of W (see Exercise 38a) and show that $V + W = \text{span}(\vec{v}_1, \ldots, \vec{v}_p, \vec{w}_1, \ldots, \vec{w}_q)$ (compare with Exercise 4).

Indeed, if $\vec{v}+\vec{w}$ is in $V+W$, then $\vec{v}$ is a linear combination of $\vec{v}_1, \ldots, \vec{v}_p$ and $\vec{w}$ is a linear combination of $\vec{w}_1, \ldots, \vec{w}_q$, so that $\vec{v} + \vec{w}$ is a linear combination of $\vec{v}_1, \ldots, \vec{v}_p, \vec{w}_1, \ldots, \vec{w}_q$. Conversely, if $\vec{x}$ is in $\text{span}(\vec{v}_1, \ldots, \vec{v}_p, \vec{w}_1, \ldots, \vec{w}_q)$, then $\vec{x} = (c_1\vec{v}_1 + \cdots + c_p\vec{v}_p) + (d_1\vec{w}_1 + \cdots + d_q\vec{w}_q)$, so that $\vec{x}$ is in $V + W$.