3.2.28 The three column vectors are linearly independent, since rref $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 5 \\ 1 & 3 & 7 \end{bmatrix} = I_3$.

Therefore, the three columns form a basis of $\text{im}(A)(= \mathbb{R}^3)$:

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ 7 \end{bmatrix}.$$

Another sensible choice for a basis of $\text{im}(A)$ is $\vec{e}_1, \vec{e}_2, \vec{e}_3$.

3.2.29 The three column vectors of A span all of $\mathbb{R}^2$, so that $\text{im}(A) = \mathbb{R}^2$. We can choose any two of the columns of A to form a basis of $\text{im}(A)$; another sensible choice is $\vec{e}_1, \vec{e}_2$.

3.2.30 $\text{im}(A) = \text{span}(\vec{e}_1, \vec{e}_2)$

We can choose $\vec{e}_1, \vec{e}_2$ as a basis of $\text{im}(A)$.

3.2.38 a Using the terminology introduced in the exercise, we need to show that any vector $\vec{v}$ in V is a linear combination of $\vec{v}_1, \ldots, \vec{v}_m$. Choose a specific vector $\vec{v}$ in V. Since we can find no more than m linearly independent vectors in V, the $m+1$ vectors $\vec{v}_1, \ldots, \vec{v}_m, \vec{v}$ will be linearly dependent. Since the vectors $\vec{v}_1, \ldots, \vec{v}_m$ are independent, $\vec{v}$ must be redundant, meaning that $\vec{v}$ is a linear combination of $\vec{v}_1, \ldots, \vec{v}_m$, as claimed.

b With the terminology introduced in part a, we can let $V = \text{im}\begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_m \end{bmatrix}$.

3.2.39 Yes; the vectors are linearly independent. The vectors in the list $\vec{v}_1, \ldots, \vec{v}_m$ are linearly independent (and therefore non-redundant), and $\vec{v}$ is non-redundant since it fails to be in the span of $\vec{v}_1, \ldots, \vec{v}_m$.

3.2.40 Yes; by Theorem 3.2.8, $\ker(A) = \{\vec{0}\}$ and $\ker(B) = \{\vec{0}\}$. Then $\ker(AB) = \{\vec{0}\}$ by Exercise 3.1.51, so that the columns of AB are linearly independent, by Theorem 3.2.8.

3.2.41 To show that the columns of B are linearly independent, we show that $\ker(B) = \{\vec{0}\}$. Indeed, if $B\vec{x} = \vec{0}$, then $AB\vec{x} = A\vec{0} = \vec{0}$, so that $\vec{x} = \vec{0}$ (since $AB = I_m$).

By Theorem 3.2.8, $\text{rank}(B) = \#\text{ columns} = m$, so that $m \le n$ and in fact $m < n$ (we are told that $m \ne n$). This implies that the rank of the $m \times n$ matrix A is less than n, so that the columns of A are linearly dependent (by Theorem 3.2.8).

3.2.42 We can use the hint and form the dot product of $\vec{v}_i$ and both sides of the relation

$c_1\vec{v}_1 + \cdots + c_i\vec{v}_i + \cdots + c_m\vec{v}_m = \vec{0}$:

$(c_1\vec{v}_1 + \cdots + c_i\vec{v}_i + \cdots + c_m\vec{v}_m) \cdot \vec{v}_i = \vec{0} \cdot \vec{v}_i$, so that $c_1(\vec{v}_1 \cdot \vec{v}_i) + \cdots + c_i(\vec{v}_i \cdot \vec{v}_i) + \cdots + c_m(\vec{v}_m \cdot \vec{v}_i) = 0$.

Since $\vec{v}_i$ is perpendicular to all the other $\vec{v}_j$, we will have $\vec{v}_i \cdot \vec{v}_j = 0$ whenever $j \ne i$; since $\vec{v}_i$ is a unit vector, we will have $\vec{v}_i \cdot \vec{v}_i = 1$. Therefore, the equation above simplifies to $c_i = 0$.

Since this reasoning applies to all $i = 1, \ldots, m$, we have only the trivial relation among the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$, so that these vectors are linearly independent, as claimed.

3.2.43 Consider a linear relation $c_1\vec{v}_1 + c_2(\vec{v}_1 + \vec{v}_2) + c_3(\vec{v}_1 + \vec{v}_2 + \vec{v}_3) = \vec{0}$, or, $(c_1 + c_2 + c_3)\vec{v}_1 + (c_2 + c_3)\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$. Since there is only the trivial relation among the vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$, we must have $c_1 + c_2 + c_3 = c_2 + c_3 = c_3 = 0$, so that $c_3 = 0$ and then $c_2 = 0$ and then $c_1 = 0$, as claimed.

3.2.44 Yes; this is a special case of Exercise 40 (recall that $\ker(A) = \{\vec{0}\}$, by Theorem 3.1.7b).

3.2.45 Yes; if A is invertible, then $\ker(A) = \{\vec{0}\}$, so that the columns of A are linearly independent, by Theorem 3.2.8.

3.2.46 Solve the system $\begin{bmatrix} x_1 & + & 2x_2 & & & + & 3x_4 & + & 5x_5 & = 0 \\ & & & & x_3 & + & 4x_4 & + & 6x_5 & = 0 \end{bmatrix}$.

The solutions are of the form

3.3.22 $\text{rref}\begin{bmatrix} 2 & 4 & 8 \\ 4 & 5 & 1 \\ 7 & 9 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -6 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{bmatrix}$. It is clear that the third vector is redundant, and we quickly see that the vector $\begin{bmatrix} -6 \\ 5 \\ -1 \end{bmatrix}$ is in the kernel. Since this is the only redundant column, $\left(\begin{bmatrix} -6 \\ 5 \\ -1 \end{bmatrix}\right)$ is a basis of the kernel. Thus, a basis of $\text{im}(A)$ is $\left(\begin{bmatrix} 2 \\ 4 \\ 7 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 9 \end{bmatrix}\right)$.

3.3.26 a We notice that each of the six matrices has two identical columns. In matrices C and L, the second column is identical to the third, so that $\ker(C) = \ker(L) = \text{span}\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$. In matrices H, T, X and Y, the first column is identical to the third, so that $\ker(H) = \ker(T) = \ker(X) = \ker(Y) = \text{span}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$. Thus, only L has the same kernel as C.

b We observe that each of the six matrices in the list has two identical rows. For example, the first and the last row of matrix C are identical, so that any vector $\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ in $\text{im}(C)$ will satisfy the equation $y_1 = y_3$. We can conclude that $\text{im}(C) = \text{im}(H) = \text{im}(X) = \left\{ \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} : y_1 = y_3 \right\}$, $\text{im}(L) = \left\{ \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} : y_1 = y_2 \right\}$, and $\text{im}(T) = \text{im}(Y) = \left\{ \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} : y_2 = y_3 \right\}$.

c Our discussion in part b shows that the answer is matrix L.

3.3.28 Form a 4×4 matrix A with the given vectors as its columns. The matrix A reduces to

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & k-29 \end{bmatrix}.$$

This matrix can be reduced further to I_4 if (and only if) $k - 29 \neq 0$, that is, if $k \neq 29$.

By Summary 3.3.10, the four given vectors form a basis of $\mathbb{R}^4$ unless $k = 29$.

3.3.29 $x_1 = -\frac{3}{2}x_2 - \frac{1}{2}x_3$; let $x_2 = s$ and $x_3 = t$. Then the solutions are of the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\frac{3}{2}s - \frac{1}{2}t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} -\frac{3}{2} \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix}.$$

Multiplying the two vectors by 2 to simplify, we obtain the basis $\begin{bmatrix} -3 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$.

3.3.30 Proceeding as in Exercise 29, we find the basis $\begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

3.3.31 Proceeding as in Exercise 29, we can find the following basis of V: $\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

Now let A be the 4×3 matrix with these three vectors as its columns. Then $\text{im}(A) = V$ by Theorem 3.1.3, and $\ker(A) = \{\vec{0}\}$ by Theorem 3.2.8, so that A does the job.

$$A = \begin{bmatrix} 1 & -2 & -4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

3.3.32 We need to find all vectors $\vec{x}$ in $\mathbb{R}^4$ such that $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} = 0$ and $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} = 0$.

This amounts to solving the system $\left[\begin{array}{ccccccc} x_1 & & - & x_3 & + & x_4 & = 0 \\ & x_2 & + & 2x_3 & + & 3x_4 & = 0 \end{array}\right]$, which in turn amounts to finding the kernel of $\begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix}$.

Using Kyle Numbers, we find the basis $\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 0 \\ 1 \end{bmatrix}$.

3.3.33 We can write $V = \ker(A)$, where A is the $1 \times n$ matrix $A = [c_1 \; c_2 \; \cdots \; c_n]$.

Since at least one of the c_i is nonzero, the rank of A is 1, so that $\dim(V) = \dim(\ker(A)) = n - \operatorname{rank}(A) = n - 1$, by Theorem 3.3.7.

A "hyperplane" in $\mathbb{R}^2$ is a line, and a "hyperplane" in $\mathbb{R}^3$ is just a plane.

3.3.34 We can write $V = \ker(A)$, where A is the $n \times m$ matrix with entries a_{ij}. Note that $\operatorname{rank}(A) \leq n$. Therefore, $\dim(V) = \dim(\ker(A)) = m - \operatorname{rank}(A) \geq m - n$, by Theorem 3.3.7.

3.3.35 We need to find all vectors $\vec{x}$ in $\mathbb{R}^n$ such that $\vec{v} \cdot \vec{x} = 0$, or $v_1x_1 + v_2x_2 + \cdots + v_nx_n = 0$, where the v_i are the components of the vector $\vec{v}$. These vectors form a hyperplane in $\mathbb{R}^n$ (see Exercise 33), so that the dimension of the space is $n - 1$.

3.3.36 No; if $\operatorname{im}(A) = \ker(A)$ for an $n \times n$ matrix A, then $n = \dim(\ker(A)) + \dim(\operatorname{im}(A)) = 2\dim(\operatorname{im}(A))$, so that n is an even number.

3.3.37 Since $\dim(\ker(A)) = 5 - \operatorname{rank}(A)$, any 4×5 matrix with rank 2 will do; for example,

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

3.3.38 a The rank of a 3×5 matrix A is 0,1,2, or 3, so that $\dim(\ker(A)) = 5 - \operatorname{rank}(A)$ is 2,3,4, or 5.

b The rank of a 7×4 matrix A is at most 4, so that $\dim(\operatorname{im}(A)) = \operatorname{rank}(A)$ is 0,1,2,3, or 4.

3.3.60 We can choose a basis $\vec{v}_1, \ldots, \vec{v}_p$ in V, where $p = \dim(V)$. Then $\vec{v}_1, \ldots, \vec{v}_p$ are linearly independent vectors in W, so that $\dim(V) = p \leq \dim(W)$, by Theorem 3.3.4a, as claimed.

3.3.61 We can choose a basis $\vec{v}_1, \ldots, \vec{v}_p$ of V, where $p = \dim(V) = \dim(W)$. Then $\vec{v}_1, \ldots, \vec{v}_p$ is a basis of W as well, by Theorem 3.3.4c, so that $V = W = \operatorname{span}(\vec{v}_1, \ldots, \vec{v}_p)$, as claimed.

3.3.62 Consider a basis $\vec{v}_1, \ldots, \vec{v}_n$ of V. Since the $\vec{v}_i$ are n linearly independent vectors in $\mathbb{R}^n$, they form a basis of $\mathbb{R}^n$ (by parts (vii) and (ix) of Summary 3.3.10), so that $V = \operatorname{span}(\vec{v}_1, \ldots, \vec{v}_n) = \mathbb{R}^n$, as claimed. (Note that Exercise 3.3.62 is a special case of Exercise 3.3.61.)

3.3.63 $\dim(V + W) = \dim(V) + \dim(W)$, by Exercise 3.2.51b.

3.3.64 Suppose that $V \cap W = \{\vec{0}\}$ and $\dim(V) + \dim(W) = n$.

Choose a basis $\vec{v}_1, \ldots, \vec{v}_p$ of V and a basis $\vec{w}_1, \ldots, \vec{w}_q$ in W; note that $p + q = n$. By Exercise 3.2.51b, the n vectors $\vec{v}_1, \ldots, \vec{v}_p, \vec{w}_1, \ldots, \vec{w}_q$ in $\mathbb{R}^n$ are linearly independent, so that they form a basis of $\mathbb{R}^n$ (by parts (vii) and (ix) of Summary 3.3.10). By Theorem 3.2.10, any vector $\vec{x}$ can be written uniquely as

$\vec{x} = (c_1\vec{v}_1 + \cdots + c_p\vec{v}_p) + (d_1\vec{w}_1 + \cdots + d_q\vec{w}_q)$, with $\vec{v} = c_1\vec{v}_1 + \cdots + c_p\vec{v}_p$ in V and $\vec{w} = d_1\vec{w}_1 + \cdots + d_q\vec{w}_q$ in W, which gives the desired representation.

Conversely, suppose V and W are complements. Let us first show that $V \cap W = \{\vec{0}\}$ in this case.

Indeed, if $\vec{x}$ is in $V \cap W$, then we can write

$$\underset{\text{in } V}{\vec{x}} = \underset{\text{in } W}{\vec{x}} + \underset{\text{in } V}{\vec{0}} = \underset{\text{in } W}{\vec{0}} + \vec{x}$$

Since this representation is unique (by definition of complements), we must have $\vec{x} = \vec{0}$, so that $V \cap W = \{\vec{0}\}$. By definition of complements, we have $\mathbb{R}^n = V + W$, so that $n = \dim(V + W) = \dim(V) + \dim(W)$, by Exercise 63.

3.3.65 Note that $\operatorname{im}(A) = \operatorname{span}(\vec{v}_1, \ldots, \vec{v}_p, \vec{w}_1, \ldots, \vec{w}_q) = V$, since the $\vec{w}_j$ alone span V.

To find a basis of $V = \operatorname{im}(A)$, we omit the redundant vectors from the list $\vec{v}_1, \ldots, \vec{v}_p, \vec{w}_1, \ldots \vec{w}_q$, by Theorem 3.2.4. Since the vectors $\vec{v}_1, \ldots, \vec{v}_p$ are linearly independent, none of them are redundant, so that our basis of V contains all vectors $\vec{v}_1, \ldots, \vec{v}_p$ and some of the vectors from the list $\vec{w}_1, \ldots, \vec{w}_q$.

3.3.66 Use Exercise 65 with $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 4 \\ 6 \\ 8 \end{bmatrix}$, and $\vec{w}_i = \vec{e}_i$ for $i = 1, 2, 3, 4$.

Now $\operatorname{rref}\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 2 & 4 & 0 & 1 & 0 & 0 \\ 3 & 6 & 0 & 0 & 1 & 0 \\ 4 & 8 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 & 0 & 0 & -\frac{1}{4} \\ 0 & 1 & -1 & 0 & 0 & \frac{1}{4} \\ 0 & 0 & 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 & -\frac{3}{4} \end{bmatrix}$.

Picking the non-redundant columns gives the basis $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 6 \\ 8 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$.

3.3.82 Write $A = [\vec{v}_1 \ \dots \ \vec{v}_m]$ and $B = [\vec{w}_1 \ \dots \ \vec{w}_m]$, so that $A+B = [\vec{v}_1 + \vec{w}_1 \ \cdots \vec{v}_m + \vec{w}_m]$. Any linear combination of the columns of $A + B$, $\vec{y} = c_1(\vec{v}_1 + \vec{w}_1) + \cdots + c_m(\vec{v}_m + \vec{w}_m)$, can be written as

$$\vec{y} = \underbrace{(c_1\vec{v} + \cdots + c_m\vec{v}_m)}_{\text{in im}(A)} + \underbrace{(c_1\vec{w}_1 + \cdots + c_m\vec{w}_m)}_{\text{in im}(B)}$$

so that $\text{im}(A + B) \subseteq \text{im}(A) + \text{im}(B)$ (see Exercise 3.2.50). Since $\dim(V + W) \le \dim(V) + \dim(W)$, by Exercise 3.3.67, we can conclude that $\text{rank}(A+B) = \dim(\text{im}(A+B)) \le \dim(\text{im}(A)) + \dim(\text{im}(B)) = \text{rank}(A) + \text{rank}(B)$.

Summary: $\text{rank}(A + B) \le \text{rank}(A) + \text{rank}(B)$.