Section 3.4

3.4.1 $\begin{bmatrix} 2 \\ 3 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, so $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

3.4.2 $\begin{bmatrix} 3 \\ -4 \end{bmatrix} = -4\begin{bmatrix} 0 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, so $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$.

3.4.3 $\begin{bmatrix}31\\37\end{bmatrix} = 0\begin{bmatrix}23\\29\end{bmatrix} + 1\begin{bmatrix}31\\37\end{bmatrix}$, so $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix}0\\1\end{bmatrix}$.

3.4.4 $\begin{bmatrix}23\\29\end{bmatrix} = \frac{1}{2}\begin{bmatrix}46\\58\end{bmatrix} + 0\begin{bmatrix}61\\67\end{bmatrix}$, so $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix}\frac{1}{2}\\0\end{bmatrix}$.

3.4.5 $\begin{bmatrix}7\\16\end{bmatrix} = -4\begin{bmatrix}2\\5\end{bmatrix} + 3\begin{bmatrix}5\\12\end{bmatrix}$, so $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix}-4\\3\end{bmatrix}$.

This may not be as obvious as Exercises 1 and 3, but we can find our coefficients simply by reducing the matrix $\left[\begin{array}{cc|c}2 & 5 & 7\\5 & 12 & 16\end{array}\right]$.

3.4.6 $\begin{bmatrix}-4\\4\end{bmatrix} = 11\begin{bmatrix}1\\2\end{bmatrix} - 3\begin{bmatrix}5\\6\end{bmatrix}$, so $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix}11\\-3\end{bmatrix}$.

We arrive at this solution by reducing the matrix $\left[\begin{array}{cc|c}1 & 5 & -4\\2 & 6 & 4\end{array}\right]$.

3.4.7 We need to find the scalars c_1 and c_2 such that $\begin{bmatrix}3\\1\\-4\end{bmatrix} = c_1\begin{bmatrix}1\\-1\\0\end{bmatrix} + c_2\begin{bmatrix}0\\1\\-1\end{bmatrix}$. Solving a linear system gives $c_1 = 3,\ c_2 = 4$. Thus $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix}c_1\\c_2\end{bmatrix} = \begin{bmatrix}3\\4\end{bmatrix}$.

3.4.8 We need to find the scalars c_1 and c_2 such that $\begin{bmatrix}2\\3\\4\end{bmatrix} = c_1\begin{bmatrix}1\\1\\0\end{bmatrix} + c_2\begin{bmatrix}2\\0\\1\end{bmatrix}$. Attempting to solve the linear system reveals an inconsistency; $\vec{x}$ is not in the span of $\vec{v}_1$ and $\vec{v}_2$.

3.4.9 We can solve this by inspection: Note that our first coefficient must be 3 because of the first terms of the vectors. Also, the second coefficient must be 2 due to the last terms.

However, $3\vec{v}_1 + 2\vec{v}_2 = \begin{bmatrix}3\\3\\0\end{bmatrix} + \begin{bmatrix}0\\-2\\4\end{bmatrix} = \begin{bmatrix}3\\1\\4\end{bmatrix}$. Thus, we reason that $\vec{x}$ is not in the span of $\vec{v}_1$ and $\vec{v}_2$.

We can also see this by attempting to solve $\left[\begin{array}{cc|c}1 & 0 & 3\\1 & -1 & 3\\0 & 2 & 4\end{array}\right]$, which turns out to be inconsistent. Thus, $\vec{x}$ is not in V.

3.4.10 Proceeding as in Example 1, we find $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix}3\\1\end{bmatrix}$.

3.4.11 Proceeding as in Example 1, we find $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\end{bmatrix}$.

3.4.12 Proceeding as in Example 1, we find $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix}-3\\5\end{bmatrix}$.

$$\vec{x} = c_1\begin{bmatrix}1\\1\end{bmatrix} + c_2\begin{bmatrix}1\\-1\end{bmatrix} \quad \xrightarrow{T} \quad T(\vec{x}) = A\vec{x} = c_1A\begin{bmatrix}1\\1\end{bmatrix} + c_2A\begin{bmatrix}1\\-1\end{bmatrix}$$
$$= c_1\begin{bmatrix}1\\1\end{bmatrix} + c_2\begin{bmatrix}-1\\1\end{bmatrix} = c_1\begin{bmatrix}1\\1\end{bmatrix} - c_2\begin{bmatrix}1\\-1\end{bmatrix}$$
$$\downarrow \qquad\qquad\qquad \downarrow$$
$$[\vec{x}]_{\mathcal{B}} = \begin{bmatrix}c_1\\c_2\end{bmatrix} \quad \xrightarrow{B} \quad [T(\vec{x})]_{\mathcal{B}} = \begin{bmatrix}c_1\\-c_2\end{bmatrix}$$

So, $\begin{bmatrix}a & b\\c & d\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix} = \begin{bmatrix}c_1\\-c_2\end{bmatrix}$, and we quickly find $B = \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$.

c $B = [[T(\vec{v}_1)]_{\mathcal{B}}[T(\vec{v}_2)]_{\mathcal{B}}] = \left[\left[\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}\right]_{\mathcal{B}} \left[\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix}\right]_{\mathcal{B}}\right] = \left[\begin{bmatrix}1\\1\end{bmatrix}_{\mathcal{B}}\begin{bmatrix}-1\\1\end{bmatrix}_{\mathcal{B}}\right] = \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$.

3.4.20 a $S = \begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}$, and we find the inverse S^{-1} to be equal to $\frac{1}{2}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}$.

Then $B = S^{-1}AS = \frac{1}{2}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}2 & 2\\0 & 0\end{bmatrix}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}4 & 0\\0 & 0\end{bmatrix} = \begin{bmatrix}2 & 0\\0 & 0\end{bmatrix}$.

b Our commutative diagram:

$$\vec{x} = c_1\begin{bmatrix}1\\1\end{bmatrix} + c_2\begin{bmatrix}1\\-1\end{bmatrix} \quad \xrightarrow{T} \quad T(\vec{x}) = A\vec{x} = c_1A\begin{bmatrix}1\\1\end{bmatrix} + c_2A\begin{bmatrix}1\\-1\end{bmatrix}$$
$$= c_1\begin{bmatrix}2\\2\end{bmatrix} + c_2\begin{bmatrix}0\\0\end{bmatrix} = 2c_1\begin{bmatrix}1\\1\end{bmatrix} + 0\begin{bmatrix}1\\-1\end{bmatrix}$$
$$\downarrow \qquad\qquad\qquad \downarrow$$
$$[\vec{x}]_{\mathcal{B}} = \begin{bmatrix}c_1\\c_2\end{bmatrix} \quad \xrightarrow{B} \quad [T(\vec{x})]_{\mathcal{B}} = \begin{bmatrix}2c_1\\0\end{bmatrix}$$

So, $\begin{bmatrix}a & b\\c & d\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix} = \begin{bmatrix}2c_1\\0\end{bmatrix}$, and we quickly find $B = \begin{bmatrix}2 & 0\\0 & 0\end{bmatrix}$.

c $B = [[T(\vec{v}_1)]_{\mathcal{B}}[T(\vec{v}_2)]_{\mathcal{B}}] = \left[\left[\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}\right]_{\mathcal{B}} \left[\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix}\right]_{\mathcal{B}}\right] = \left[\begin{bmatrix}2\\2\end{bmatrix}_{\mathcal{B}}\begin{bmatrix}0\\0\end{bmatrix}_{\mathcal{B}}\right] = \begin{bmatrix}2 & 0\\0 & 0\end{bmatrix}$.

3.4.21 a $S = \begin{bmatrix}1 & -2\\3 & 1\end{bmatrix}$, and we find the inverse S^{-1} to be equal to $\frac{1}{7}\begin{bmatrix}1 & 2\\-3 & 1\end{bmatrix}$.

Then $B = S^{-1}AS = \frac{1}{7}\begin{bmatrix}1 & 2\\-3 & 1\end{bmatrix}\begin{bmatrix}1 & 2\\3 & 6\end{bmatrix}\begin{bmatrix}1 & -2\\3 & 1\end{bmatrix} = \frac{1}{7}\begin{bmatrix}7 & 14\\0 & 0\end{bmatrix}\begin{bmatrix}1 & -2\\3 & 1\end{bmatrix} = \frac{1}{7}\begin{bmatrix}49 & 0\\0 & 0\end{bmatrix} = \begin{bmatrix}7 & 0\\0 & 0\end{bmatrix}$.

b Our commutative diagram:

b Our commutative diagram:

$$\vec{x} = c_1\begin{bmatrix}1\\1\end{bmatrix} + c_2\begin{bmatrix}1\\2\end{bmatrix} \quad \overrightarrow{T} \quad T(\vec{x}) = A\vec{x} = c_1A\begin{bmatrix}1\\1\end{bmatrix} + c_2A\begin{bmatrix}1\\2\end{bmatrix}$$

$$= c_1\begin{bmatrix}2\\2\end{bmatrix} + c_2\begin{bmatrix}-1\\-2\end{bmatrix} = 2c_1\begin{bmatrix}1\\1\end{bmatrix} - 1c_2\begin{bmatrix}1\\2\end{bmatrix}$$

$$\downarrow \qquad\qquad \downarrow$$

$$[\vec{x}]_{\mathcal{B}} = \begin{bmatrix}c_1\\c_2\end{bmatrix} \quad \overrightarrow{B} \quad [T(\vec{x})]_{\mathcal{B}} = \begin{bmatrix}2c_1\\-c_2\end{bmatrix}$$

So, $\begin{bmatrix}a & b\\c & d\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix} = \begin{bmatrix}2c_1\\-c_2\end{bmatrix}$, and we quickly find $B = \begin{bmatrix}2 & 0\\0 & -1\end{bmatrix}$.

c $B = [[T(\vec{v}_1)]_{\mathcal{B}}[T(\vec{v}_2)]_{\mathcal{B}}] = \left[\left[\begin{bmatrix}5 & -3\\6 & -4\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}\right]_{\mathcal{B}} \left[\begin{bmatrix}5 & -3\\6 & -4\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix}\right]_{\mathcal{B}}\right]$

$= \left[\begin{bmatrix}2\\2\end{bmatrix}_{\mathcal{B}} \begin{bmatrix}-1\\-2\end{bmatrix}_{\mathcal{B}}\right] = \begin{bmatrix}2 & 0\\0 & -1\end{bmatrix}$.

3.4.**24** a $S = \begin{bmatrix}2 & 5\\1 & 3\end{bmatrix}$, and we find the inverse S^{-1} to be equal to $\begin{bmatrix}3 & -5\\-1 & 2\end{bmatrix}$.

Then $B = S^{-1}AS = \begin{bmatrix}3 & -5\\-1 & 2\end{bmatrix}\begin{bmatrix}13 & -20\\6 & -9\end{bmatrix}\begin{bmatrix}2 & 5\\1 & 3\end{bmatrix} = \begin{bmatrix}9 & -15\\-1 & 2\end{bmatrix}\begin{bmatrix}2 & 5\\1 & 3\end{bmatrix} = \begin{bmatrix}3 & 0\\0 & 1\end{bmatrix}$.

b Our commutative diagram:

$$\vec{x} = c_1\begin{bmatrix}2\\1\end{bmatrix} + c_2\begin{bmatrix}5\\3\end{bmatrix} \quad \overrightarrow{T} \quad T(\vec{x}) = A\vec{x} = c_1A\begin{bmatrix}2\\1\end{bmatrix} + c_2A\begin{bmatrix}5\\3\end{bmatrix}$$

$$= c_1\begin{bmatrix}6\\3\end{bmatrix} + c_2\begin{bmatrix}5\\3\end{bmatrix} = 3c_1\begin{bmatrix}2\\1\end{bmatrix} + 1c_2\begin{bmatrix}5\\3\end{bmatrix}$$

$$\downarrow \qquad\qquad \downarrow$$

$$[\vec{x}]_{\mathcal{B}} = \begin{bmatrix}c_1\\c_2\end{bmatrix} \quad \overrightarrow{B} \quad [T(\vec{x})]_{\mathcal{B}} = \begin{bmatrix}3c_1\\c_2\end{bmatrix}$$

So, $\begin{bmatrix}a & b\\c & d\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix} = \begin{bmatrix}3c_1\\c_2\end{bmatrix}$, and we quickly find $B = \begin{bmatrix}3 & 0\\0 & 1\end{bmatrix}$.

c $B = [[T(\vec{v}_1)]_{\mathcal{B}}[T(\vec{v}_2)]_{\mathcal{B}}] = \left[\left[\begin{bmatrix}13 & -20\\6 & -9\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix}\right]_{\mathcal{B}} \left[\begin{bmatrix}13 & -20\\6 & -9\end{bmatrix}\begin{bmatrix}5\\3\end{bmatrix}\right]_{\mathcal{B}}\right]$

$= \left[\begin{bmatrix}6\\3\end{bmatrix}_{\mathcal{B}} \begin{bmatrix}5\\3\end{bmatrix}_{\mathcal{B}}\right] = \begin{bmatrix}3 & 0\\0 & 1\end{bmatrix}$.

3.4.**25** We will use the commutative diagram method here (though any method suffices).

3.4.**28** Let's build B "column-by-column":

$$B = [[T(\vec{v}_1)]_{\mathcal{B}}[T(\vec{v}_2)]_{\mathcal{B}}[T(\vec{v}_3)]_{\mathcal{B}}]$$

$$= \left[\left[\begin{bmatrix} 5 & -4 & -2 \\ -4 & 5 & -2 \\ -2 & -2 & 8 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \right]_{\mathcal{B}} \; \left[\begin{bmatrix} 5 & -4 & -2 \\ -4 & 5 & -2 \\ -2 & -2 & 8 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right]_{\mathcal{B}} \; \left[\begin{bmatrix} 5 & -4 & -2 \\ -4 & 5 & -2 \\ -2 & -2 & 8 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} \right]_{\mathcal{B}} \right]$$

$$= \left[\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}_{\mathcal{B}} \; \begin{bmatrix} 9 \\ -9 \\ 0 \end{bmatrix}_{\mathcal{B}} \; \begin{bmatrix} 0 \\ 9 \\ -18 \end{bmatrix}_{\mathcal{B}} \right] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}.$$

3.4.40 From Exercise 37, we see that we want one of our basis vectors to be parallel to the line, while the others must be perpendicular the line. We can easily find such a basis: $\mathcal{B} = \left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right)$.

3.4.46 As in Exercise 3.4.45, we can make $\vec{v}_1$ any vector in the plane that is not parallel to $\vec{x}$, and then let $\vec{v}_2 = 2\vec{v}_1 - \vec{x}$.

For example, if we choose $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, then $\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}$.

3.4.47 By Theorem 3.4.4, we have $A = SBS^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} d & c \\ b & a \end{bmatrix}$.

3.4.48 $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ means that $\vec{x} = -\vec{v} + 2\vec{w}$. See Figure 3.6

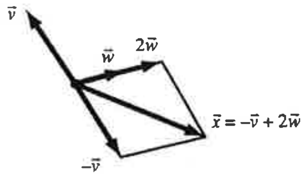

Figure 3.6: for Problem 3.4.48.

3.4.49 $\vec{u} + \vec{v} = -\vec{w}$, so that $\vec{w} = -\vec{u} - \vec{v}$, i.e., $[\vec{w}]_{\mathcal{B}} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$.

3.4.50 a $\overrightarrow{OP} = \vec{w} + 2\vec{v}$, so that $[\overrightarrow{OP}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\overrightarrow{OQ} = \vec{v} + 2\vec{w}$, so that $[\overrightarrow{OQ}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

b $\overrightarrow{OR} = 3\vec{v} + 2\vec{w}$. See Figure 3.7.

3.4.54 Let Q be the matrix whose columns are the vectors of the basis $\mathcal{T}$. Then $[[\vec{v}_1]_\mathcal{T} \dots [\vec{v}_n]_\mathcal{T}] = [Q^{-1}\vec{v}_1 \dots Q^{-1}\vec{v}_n] = Q^{-1}[\vec{v}_1 \dots \vec{v}_n]$ is an invertible matrix, so that the vectors $[\vec{v}_1]_\mathcal{T} \dots [\vec{v}_n]_\mathcal{T}$ form a basis of $\mathbb{R}^n$.

3.4.55 By Definition 3.4.1, we have $\vec{x} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} [\vec{x}]_\mathcal{B}$ and $\vec{x} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} [\vec{x}]_\mathcal{R}$, so that $\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} [\vec{x}]_\mathcal{B} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} [\vec{x}]_\mathcal{R}$ and

$$[\vec{x}]_\mathcal{R} = \underbrace{\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}}_{P} [\vec{x}]_\mathcal{B}, \text{ i.e., } P = \begin{bmatrix} -\frac{1}{2} & 1 \\ \frac{1}{2} & 0 \end{bmatrix}.$$

3.4.56 Let $S = [\vec{v}_1 \vec{v}_2]$ where $\vec{v}_1, \vec{v}_2$ is the desired basis. Then by Theorem 3.4.1, $\begin{bmatrix} 1 \\ 2 \end{bmatrix} = S \begin{bmatrix} 3 \\ 5 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 4 \end{bmatrix} = S \begin{bmatrix} 2 \\ 3 \end{bmatrix}$,

i.e. $S \begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$. Hence $S = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 12 & -7 \\ 14 & -8 \end{bmatrix}$. The desired basis is $\begin{bmatrix} 12 \\ 14 \end{bmatrix}, \begin{bmatrix} -7 \\ -8 \end{bmatrix}$.

3.4.60 First we find the matrices $S = \begin{bmatrix} x & y \\ z & t \end{bmatrix}$ such that $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x & y \\ z & t \end{bmatrix} = \begin{bmatrix} x & y \\ z & t \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, or, $\begin{bmatrix} x & y \\ -z & -t \end{bmatrix} = \begin{bmatrix} y & x \\ t & z \end{bmatrix}$. The solutions are of the form $S = \begin{bmatrix} y & y \\ -t & t \end{bmatrix}$, where y and t are arbitrary constants. Since there are *invertible* solutions S (for example, let $y = t = 1$), the matrices $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ are indeed similar.

3.4.61 We seek a basis $\vec{v}_1 = \begin{bmatrix} x \\ z \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} y \\ t \end{bmatrix}$ such that the matrix $S = [\vec{v}_1 \ \vec{v}_2] = \begin{bmatrix} x & y \\ z & t \end{bmatrix}$ satisfies the equation $\begin{bmatrix} -5 & -9 \\ 4 & 7 \end{bmatrix}\begin{bmatrix} x & y \\ z & t \end{bmatrix} = \begin{bmatrix} x & y \\ z & t \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Solving the ensuing linear system gives $S = \begin{bmatrix} -\frac{3z}{2} & \frac{z}{4} - \frac{3t}{2} \\ z & t \end{bmatrix}$.

We need to choose z and t so that S will be invertible. For example, if we let $z = 6$ and $t = 1$, then $S = \begin{bmatrix} -9 & 0 \\ 6 & 1 \end{bmatrix}$, so that $\vec{v}_1 = \begin{bmatrix} -9 \\ 6 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

3.4.70 Suppose such a basis $\vec{v}_1, \vec{v}_2$ exists. If $B = [[T(\vec{v}_1)]_{\mathcal{B}}\,[T(\vec{v}_2)]_{\mathcal{B}}]$ is upper triangular, of the form $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$, then $[T(\vec{v}_1)]_{\mathcal{B}} = \begin{bmatrix} a \\ 0 \end{bmatrix}$, so that $T(\vec{v}_1) = a\vec{v}_1$, that is, $T(\vec{v}_1)$ is parallel to $\vec{v}_1$. But this is impossible, since T is a rotation through $\frac{\pi}{2}$.

3.4.71 a Note that $AS = SB$. If $\vec{x}$ is in $\ker(B)$, then $A(S\vec{x}) = SB\vec{x} = S\vec{0} = \vec{0}$, so that $S\vec{x}$ is in $\ker(A)$, as claimed.

b We use the hint and observe that nullity $(B) = \dim(\ker B) = p \leq \dim(\ker A) = \text{nullity}(A)$, since $S\vec{v}_1, \ldots, S\vec{v}_p$ are p linearly independent vectors in $\ker(A)$. Reversing the roles of A and B shows that, conversely, $\text{nullity}(A) \leq \text{nullity}(B)$, so that the equation $\text{nullity}(A) = \text{nullity}(B)$ holds, as claimed.

3.4.72 If A and B are similar $n \times n$ matrices, then $\text{rank}(A) = n - \text{nullity}(A) = n - \text{nullity}(B) = \text{rank}(B)$, by Exercise 71 and the rank nullity theorem (Theorem 3.3.7).