3.1.52 Since $C\vec{x} = \begin{bmatrix} A \\ B \end{bmatrix} \vec{x} = \begin{bmatrix} A\vec{x} \\ B\vec{x} \end{bmatrix}$, we can conclude that $C\vec{x} = \vec{0}$ if (and only if) both $A\vec{x} = \vec{0}$ and $B\vec{x} = \vec{0}$. It follows that $\ker(C)$ is the intersection of $\ker(A)$ and $\ker(B)$: $\ker(C) = \ker(A) \cap \ker(B)$.

3.1.53 a Using the equation $1 + 1 = 0$ (or $-1 = 1$), we can write the general vector $\vec{x}$ in $\ker(H)$ as

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} p+r+s \\ p+q+s \\ p+q+r \\ p \\ q \\ r \\ s \end{bmatrix}$$

$$= p \underset{\vec{v}_1}{\underset{\uparrow}{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}} + q \underset{\vec{v}_2}{\underset{\uparrow}{\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}}} + r \underset{\vec{v}_3}{\underset{\uparrow}{\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}}} + s \underset{\vec{v}_4}{\underset{\uparrow}{\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}}}$$

b $\ker(H) = \text{span}(\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4)$ by part (a), and $\text{im}(M) = \text{span}(\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4)$ by Theorem 3.1.3, so that $\text{im}(M) = \ker(H)$. $M\vec{x}$ is in $\text{im}(M) = \ker(H)$, so that $H(M\vec{x}) = \vec{0}$.

3.1.54 a If no error occurred, then $\vec{w} = \vec{v} = M\vec{u}$, and $H\vec{w} = H(M\vec{u}) = \vec{0}$, by Exercise 53b.

If an error occurred in the ith component, then $\vec{w} = \vec{v} + \vec{e}_i = M\vec{u} + \vec{e}_i$, so that

$H\vec{w} = H(M\vec{u}) + H\vec{e}_i = i$th column of H.

Since the columns of H are all different, this method allows us to find out where an error occurred.

b $H\vec{w} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ = seventh column of H: an error occurred in the seventh component of $\vec{v}$.

Therefore $\vec{v} = \vec{w} + \vec{e}_7 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$ and $\vec{u} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$.

Section 3.2

3.2.1 Not a subspace, since W does not contain the zero vector.

Chapter 4

Section 4.1

4.1.1 Not a subspace since it does not contain the neutral element, that is, the function $f(t) = 0$, for all t.

4.1.2 This subset V is a subspace of P_2:

- The neutral element $f(t) = 0$ (for all t) is in V.
- If f and g are in V (so that $f(2) = g(2) = 0$), then $(f + g)(2) = f(2) + g(2) = 0 + 0 = 0$, so that $f + g$ is in V.
- If f is in V (so that $f(2) = 0$), and k is any constant, then $(kf)(2) = kf(2) = 0$, so that kf is in V.

A polynomial $f(t) = a + bt + ct^2$ is in V if $f(2) = a + 2b + 4c = 0$, or $a = -2b - 4c$. The general element of V is of the form $f(t) = (-2b - 4c) + bt + ct^2 = b(t - 2) + c(t^2 - 4)$, so that $t - 2$, $t^2 - 4$ is a basis of V.

4.1.8 This is a subspace; the justification is analogous to Exercise 7.

4.1.9 Not a subspace; consider multiplication with a negative scalar. I_3 belongs to the set, but $-I_3$ doesn't.

4.1.10 a Let $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. Let V be the set of all 3×3 matrices A such that $A\vec{v} = \vec{0}$. Then V is a subspace of $\mathbb{R}^{3\times 3}$:

b The zero matrix 0 is in V, since $0\vec{v} = \vec{0}$.

c If A and B are in V, then so is $A + B$, since $(A + B)\vec{v} = A\vec{v} + B\vec{v} = \vec{0} + \vec{0} = \vec{0}$.

d If A is in V, then so is kA for all scalars k, since $(kA)\vec{v} = k(A\vec{v}) = k\vec{0} = \vec{0}$.

4.1.11 Not a subspace: I_3 is in rref, but the scalar multiple $2I_3$ isn't.

4.1.12 Yes, the set W of all arithmetic sequences is a subspace of V. Use the fact that a sequence $(x_0, x_1, x_2, \ldots)$ is arithmetic if $x_n = x_0 + kn$ for some constant k.

- The sequence $(0, 0, 0, \ldots)$ is an arithmetic sequence, with $k = 0$.

- If (x_n) and (y_n) are arithmetic sequences (with $x_n = x_0 + pn$ and $y_n = y_0 + qn$), then $x_n + y_n = x_0 + y_0 + (p+q)n$, so that $(x_n + y_n)$ is an arithmetic sequence as well.

- If (x_n) is an arithmetic sequence (with $x_n = x_0 + pn$) and k is an arbitrary constant, then $kx_n = kx_0 + (kp)n$, so that (kx_n) is an arithmetic sequence as well.

4.1.13 Not a subspace: $(1, 2, 4, 8, \ldots)$ and $(1, 1, 1, 1, \ldots)$ are both geometric sequences, but their sum $(2, 3, 5, 9, \ldots)$ is not, since the ratios of consecutive terms fail to be equal, for example, $\frac{3}{2} \neq \frac{5}{3}$.

4.1.14 Yes

- $(0, 0, 0, \ldots, 0, \ldots)$ converges to 0.
- If $\lim_{n\to\infty} x_n = 0$ and $\lim_{n\to\infty} y_n = 0$, then $\lim_{n\to\infty}(x_n + y_n) = \lim_{n\to\infty} x_n + \lim_{n\to\infty} y_n = 0$.
- If $\lim_{n\to\infty} x_n = 0$ and k is any constant, then $\lim_{n\to\infty}(kx_n) = k \lim_{n\to\infty} x_n = 0$.

4.1.15 The set W of all square-summable sequences is a subspace of V:

- The sequence $(0, 0, 0, \ldots)$ is in W.
- Suppose (x_n) and (y_n) are in W. Note that the inequality $(x_n + y_n)^2 \leq 2x_n^2 + 2y_n^2$ holds for all n, since $2x_n^2 + 2y_n^2 - (x_n + y_n)^2 = x_n^2 + y_n^2 - 2x_ny_n = (x_n - y_n)^2 \geq 0$. Thus $\sum_{n=1}^{\infty}(x_n + y_n)^2 \leq 2\sum_{n=1}^{\infty} x_n^2 + 2\sum_{n=1}^{\infty} y_n^2$ converges, so that the sequence $(x_n + y_n)$ is in W as well.
- If (x_n) is in W $\left(\text{so that } \sum_{n=1}^{\infty} x_n^2 \text{ converges}\right)$, then (kx_n) is in W as well, for any constant k, since

$$\sum_{n=1}^{\infty}(kx_n)^2 = k^2 \sum_{n=1}^{\infty} x_n^2$$

will converge.

4.1.16 $\begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} + e\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} + f\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$

The matrices $\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$ form a basis of $\mathbb{R}^{3\times 2}$, so that $\dim(\mathbb{R}^{3\times 2}) = 6$.

4.1.17 Let E_{ij} be the $n \times m$ matrix with a 1 as its ijth entry, and zeros everywhere else. Any A in $\mathbb{R}^{n\times m}$ can be written as the sum of all $a_{ij}E_{ij}$, and the E_{ij} are linearly independent, so that they form a basis of $\mathbb{R}^{n\times m}$. Thus $\dim(\mathbb{R}^{n\times m}) = nm$.

4.1.18 Any f in P_n can be written as a linear combination of $1, t, t^2, \ldots, t^n$, by definition of P_n. Also, $1, t, \ldots, t^n$ are linearly independent; to see this consider a relation $c_0 + c_1t + \cdots + c_nt^n = 0$; since the polynomial $c_0 + c_1t + \cdots + c_nt^n$ has more than n zeros, we must have $c_0 = c_1 = \cdots = c_n = 0$, as claimed. Thus, $\dim(P_n) = n + 1$.

4.1.22 Using Exercise 21 as a guide, we find the basis $E_{11}, E_{22}, \ldots, E_{nn}$, where E_{ii} is the $n \times n$ matrix with all 0 entries, except for a 1 at the ith place on the diagonal. The dimension of this space is n.

4.1.23 Proceeding as in Exercise 21, we find the basis $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$; the dimension is 3.

4.1.24 Proceeding as in Exercise 21, we find the basis $E_{11}, E_{12}, E_{13}, E_{22}, E_{23}, E_{33}$. Here E_{ij} is the 3×3 matrix with all 0 entries, except for a 1 in the ith component of the jth column; the dimension is 6.

4.1.32 We are looking for the matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$,

or, $\begin{bmatrix} a+c & b+d \\ a+c & b+d \end{bmatrix} = \begin{bmatrix} 2a & 0 \\ 2c & 0 \end{bmatrix}$. It is required that $a = c$ and $b = -d$.

The general element is $\begin{bmatrix} a & b \\ a & -b \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$. Thus $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$ is a basis, and the dimension is 2.

4.1.33 Let $S = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, meaning

4.1.42 Let B be a matrix such that dim(ker(B))$= k$. Then, it is required that the columns of A contain only vectors in the kernel of B. Thus, each column of A can be written as: $c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k$, where the vectors $\vec{v}_i$ form as basis of the kernel of B. Thus, each of the n columns in A involves k arbitrary constants, and matrix A involves nk arbitrary constants overall. The space of matrices A has dimension nk, where k is an integer in the range $[0, n]$.

4.1.54 We can adapt the answer to Exercise 3.2.38a. Let m be the largest number of linearly independent elements we can find in W; note that $m \leq n$, by Exercise 53. Choose linearly independent elements $f_1, \ldots, f_m$ in W. We claim that the elements $f_1, \ldots, f_m$ span W. Indeed, if f is any element of W, then the $m+1$ elements $f_1, \ldots, f_m, f$ are linearly dependent, so that f is redundant: f is a linear combination of $f_1, \ldots, f_m$. It follows that $f_1, \ldots, f_m$ is a basis of W, so that $\dim(W) = m \leq n = \dim(V)$, as claimed.

4.2.6 Let $P = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$. Transformation T is linear, since

$T(A+B) = (A+B)P = AP + BP$ equals $T(A) + T(B) = AP + BP$, and

$T(kA) = (kA)P = kAP$ equals $kT(A) = kAP$.

No, T isn't an isomorphism, since $\ker(T) \neq \{0\}$; the matrix $A = \begin{bmatrix} 3 & -1 \\ 0 & 0 \end{bmatrix}$ is in the kernel of T (see Theorem 4.2.4a).

4.2.20 Linear, since $T((x+iy)+(z+it)) = T(x+z+i(y+t)) = x+z-i(y+t)$ equals

$T(x+iy)+T(z+it) = x-iy+z-it = x+z-i(y+t)$ and

$T(k(x+iy)) = T(kx+iky) = kx-iky$ equals $kT(x+iy) = k(x-iy) = kx-iky$.

Yes, T is an isomorphism; it's its own inverse, since $T(T(x+iy)) = T(x-iy) = x+iy$.

4.2.21 Linear, since $T((x+iy)+(z+it)) = T(x+z+i(y+t)) = y+t+i(x+z)$ equals

$T(x+iy)+T(z+it) = y+ix+t+iz = y+t+i(x+z)$, and

$T(k(x+iy)) = T(kx+iky) = ky+ikx$ equals $kT(x+iy) = k(y+ix) = ky+ikx$.

Yes, T is an isomorphism; it's its own inverse, since $T(T(x+iy)) = T(y+ix) = x+iy$.

4.2.34 Linear, since $T((x_0, x_1, x_2, \ldots) + (y_0, y_1, y_2, \ldots)) = T(x_0 + y_0, x_1 + y_1, x_2 + y_2, \ldots) =$

$(0, x_0 + y_0, x_1 + y_1, x_2 + y_2, \ldots)$ equals

$T(x_0, x_1, x_2, \ldots) + T(y_0, y_1, y_2, \ldots) = (0, x_0, x_1, x_2, \ldots) + (0, y_0, y_1, y_2, \ldots) =$

$(0, x_0 + y_0, x_1 + y_1, x_2 + y_2, \ldots)$, and

$T(k(x_0, x_1, x_2, \ldots)) = T(kx_0, kx_1, kx_2, \ldots) = (0, kx_0, kx_1, kx_2, \ldots)$ equals

$kT(x_0, x_1, x_2, \ldots) = k(0, x_0, x_1, x_2, \ldots) = (0, kx_0, kx_1, kx_2, \ldots)$.

No, T isn't an isomorphism, since $(1, 0, 0, 0, \ldots)$ isn't in $\text{im}(T)$.

4.2.42 Linear, since $T(f(t)+g(t)) = \begin{bmatrix} f(7)+g(7) \\ f(11)+g(11) \end{bmatrix}$ equals $T(f(t))+T(g(t)) =$

$\begin{bmatrix} f(7) \\ f(11) \end{bmatrix} + \begin{bmatrix} g(7) \\ g(11) \end{bmatrix} = \begin{bmatrix} f(7)+g(7) \\ f(11)+g(11) \end{bmatrix}$, and $T(kf(t)) = \begin{bmatrix} kf(7) \\ kf(11) \end{bmatrix}$ equals

$kT(f(t)) = k\begin{bmatrix} f(7) \\ f(11) \end{bmatrix} = \begin{bmatrix} kf(7) \\ kf(11) \end{bmatrix}$.

Not an isomorphism, since domain and codomain have different dimensions.

4.2.52 We need to find the matrices $A = \begin{bmatrix} x & y \\ z & t \end{bmatrix}$ such that

$\begin{bmatrix} x & y \\ z & t \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} = \begin{bmatrix} x+3y & 2x+6y \\ z+3t & 2z+6t \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. It is required that $x = -3y$ and $z = -3t$, so that the kernel of T consists of all matrices of the form $\begin{bmatrix} -3y & y \\ -3t & t \end{bmatrix}$.

The nullity (i.e., the dimension of the kernel of T) is 2.

4.2.58 The kernel consists of all infinite sequences such that

$T(x_0, x_1, x_2, \ldots) = (0, x_0, x_1, x_2, \ldots) = (0, 0, 0, 0, \ldots)$, that is, all terms x_k must be 0.

Thus the kernel consists of the zero sequence $(0, 0, 0, \ldots)$ alone. The image

consists of all infinite sequences of the form $(0, x_0, x_1, x_2, \ldots)$.

4.2.59 To find the kernel, we solve the equation $T(f(t)) = T(a + bt + ct^2) = a + 7b + 49c = 0$. It follows that $a = -7b - 49c$, and the general element of the kernel is $(-7b - 49c) + bt + ct^2 = b(-7 + t) + c(-49 + t^2)$. Then a basis of the kernel is $-7 + t, -49 + t^2$, and the nullity of T is 2. Now the rank of T must be 1, and the image is all of $\mathbb{R}$.

4.2.60 Note that $T(a+bt+ct^2) = \begin{bmatrix} a + 7b + 49c \\ a + 11b + 121c \end{bmatrix}$. To find the kernel, solve the linear system $\begin{bmatrix} a + 7b + 49c \\ a + 11b + 121c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. The solution is $a = 77c,\ b = -18c$, so that the kernel consists of all polynomials of the form $f(t) = c(77 - 18t + t^2) = c(t - 11)(t - 7)$. You can also see directly that the quadratic polynomials $f(t)$ with $f(7) = f(11) = 0$ are of this form. The nullity is 1. The image consists of all of $\mathbb{R}^2$, so that the rank is 2.

4.2.61 The kernel consists of all polynomials $f(t)$ such that $t(f(t)) = 0$ for all t, that is, the zero polynomial $f(t) = 0$ alone. The image consists of all polynomials $g(t)$ that can be written as $g(t) = t(f(t))$, meaning that we can factor out a t. These are the polynomials with constant term 0, of the form $g(t) = a_1t + a_2t^2 + \cdots + a_nt^n$.

4.2.62 The image of this transformation consists of all polynomials, since any polynomial is the derivative of another. The kernel of this transformation consists of all constant functions, or the span of the function $f(t) = 1$.

4.2.63 This is impossible, since $\dim(P_3) = 4$ and $\dim(\mathbb{R}^3) = 3$. See Theorem 4.2.4b.

4.2.64 Consider $T(a + bt + ct^2 + dt^3) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, for example.

4.2.65 a First, we need to show that $T(A + B) = T(A) + T(B)$ for all $n \times m$ matrices A

and B, that is $(T(A + B))(\vec{v}) = (T(A) + T(B))(\vec{v})$ for all $\vec{v}$ in $\mathbb{R}^m$. Indeed

$(T(A + B))(\vec{v}) = (A + B)\vec{v} = A\vec{v} + B\vec{v}$ equals

$(T(A) + T(B))(\vec{v}) = T(A)(\vec{v}) + T(B)(\vec{v}) = A\vec{v} + B\vec{v}$. Also

$(T(kA))(\vec{v}) = (kA)\vec{v} = kA\vec{v}$ equals $(kT(A))(\vec{v}) = k(T(A))(\vec{v}) = kA\vec{v}$.

b The kernel of T consists of all $n \times m$ matrices A such that $T(A) = 0$, that is $(T(A))(\vec{v}) = A\vec{v} = \vec{0}$ for all $\vec{v}$ in $\mathbb{R}^m$. This holds for the zero matrix only. Thus $\ker(T) = \{0\}$.

c This is true by definition of a linear transformation (Definition 2.1.1).

d Note that T gives an isomorphism from $\mathbb{R}^{n\times m}$ to $L(\mathbb{R}^m, \mathbb{R}^n)$, by parts a, b, and c.

Since $\dim(\mathbb{R}^{n\times m}) = nm$, by Exercise 17 of Seciton 4.1, we have $\dim(L(\mathbb{R}^m, \mathbb{R}^n)) = nm$,

by Theorem 4.2.4b.

4.2.**66** The kernel of T consists of all smooth functions $f(t)$ such that

$T(f(t)) = f(t) - f'(t) = 0$, or $f'(t) = f(t)$. As you may recall from a

discussion of exponential functions in calculus, those are the functions of the

form $f(t) = Ce^t$, where C is a constant. Thus the nullity of T is 1.

4.2.72 We satisfy all the requirements of Definition 4.1.2. Clearly 0 is an element of Z_n. Let $h = f + g$, where f and g are elements of Z_n. Then $h(0) = f(0) + g(0) = 0 + 0 = 0$. Also, if f is in Z_n, then $kf(0) = k(0) = 0$. We notice that the space Z_n has the basis, $t, t^2, \ldots, t^n$. Thus, the dimension of Z_n is n.

Section 4.3

4.3.1 Let $\mathcal{B}$ be the standard basis of $P_2 : 1, t, t^2$. Then the coordinates of the given polynomials with respect to $\mathcal{B}$ are $[f]_{\mathcal{B}} = \begin{bmatrix} 7 \\ 3 \\ 1 \end{bmatrix}, [g]_{\mathcal{B}} = \begin{bmatrix} 9 \\ 9 \\ 4 \end{bmatrix}, [h]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$. Finding rref$\begin{bmatrix} 7 & 9 & 3 \\ 3 & 9 & 2 \\ 1 & 4 & 1 \end{bmatrix} = I_3$, we conclude that $[f]_{\mathcal{B}}, [g]_{\mathcal{B}}, [h]_{\mathcal{B}}$ are linearly independent, hence so are f, g, h, since the coordinate transformation is an isomorphism.

4.3.2 Let $\mathcal{B}$ be the basis $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ of $\mathbb{R}^{2\times 2}$. Then the coordinates of the given matrices with respect to $\mathcal{B}$ are $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} 2 \\ 3 \\ 5 \\ 7 \end{bmatrix}, \begin{bmatrix} 1 & 4 \\ 6 & 8 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} 1 \\ 4 \\ 6 \\ 8 \end{bmatrix}$. Finding rref$\begin{bmatrix} 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 5 & 6 \\ 1 & 4 & 7 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \neq I_4$, we conclude that the four vectors $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 5 \\ 7 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 6 \\ 8 \end{bmatrix}$ are linearly dependent, and so are the four given matrices. In fact $\begin{bmatrix} 1 & 4 \\ 6 & 8 \end{bmatrix} = -\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + 4\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}$.

4.3.3 We proceed as in Exercise 1. Since rref$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 7 & 8 & 8 \\ 9 & 0 & 1 & 4 \\ 1 & 7 & 5 & 8 \end{bmatrix} = I_4$, the four given polynomials do form a basis of P_3.

4.3.4 Consider the coordinate vectors of the 3 given polynomials with respect to the standard basis of P_2: $1, t, t^2$.

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2k \\ 2+k \\ 1 \end{bmatrix}$$

Since matrix $\begin{bmatrix} 1 & 0 & 2k \\ 1 & 1 & 2+k \\ 0 & 1 & 1 \end{bmatrix}$ reduces to $\begin{bmatrix} 1 & 0 & 2k \\ 0 & 1 & 2-k \\ 0 & 0 & k-1 \end{bmatrix}$, these three vectors form a basis of $\mathbb{R}^3$ unless $k = 1$.