5.1.10 $\vec{u} \cdot \vec{v} = 2 + 3k + 4 = 6 + 3k$. The two vectors enclose a right angle if $\vec{u} \cdot \vec{v} = 6 + 3k = 0$, that is, if $k = -2$.

5.1.16 You may be able to find the solutions by educated guessing. Here is the systematic approach: we first find all vectors $\vec{x}$ that are orthogonal to $\vec{v}_1, \vec{v}_2$, and $\vec{v}_3$, then we identify the unit vectors among them.

Finding the vectors $\vec{x}$ with $\vec{x} \cdot \vec{v}_1 = \vec{x} \cdot \vec{v}_2 = \vec{x} \cdot \vec{v}_3 = 0$ amounts to solving the system

$$\begin{bmatrix} x_1 + x_2 + x_3 + x_4 = 0 \\ x_1 + x_2 - x_3 - x_4 = 0 \\ x_1 - x_2 + x_3 - x_4 = 0 \end{bmatrix}$$

(we can omit all the coefficients $\frac{1}{2}$).

The solutions are of the form $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} t \\ -t \\ -t \\ t \end{bmatrix}$.

Since $\|\vec{x}\| = 2|t|$, we have a unit vector if $t = \frac{1}{2}$ or $t = -\frac{1}{2}$. Thus there are two possible choices for $\vec{v}_4$:

$$\begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \text{ and } \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}.$$

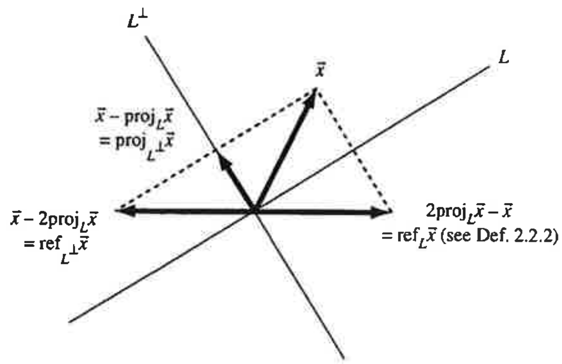

Figure 5.2: for Problem 5.1.19.

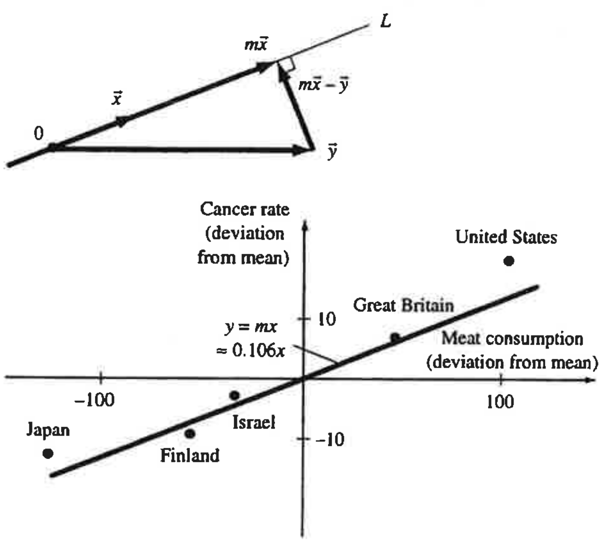

Figure 5.3: for Problem 5.1.20.

5.1.**21** Call the three given vectors $\vec{v}_1, \vec{v}_2$, and $\vec{v}_3$. Since $\vec{v}_2$ is required to be a unit vector, we must have $b = g = 0$. Now $\vec{v}_1 \cdot \vec{v}_2 = d$ must be zero, so that $d = 0$.

Likewise, $\vec{v}_2 \cdot \vec{v}_3 = e$ must be zero, so that $e = 0$.

Since $\vec{v}_3$ must be a unit vector, we have $\|\vec{v}_3\|^2 = c^2 + \frac{1}{4} = 1$, so that $c = \pm\frac{\sqrt{3}}{2}$.

Since we are asked to find just one solution, let us pick $c = \frac{\sqrt{3}}{2}$.

The condition $\vec{v}_1 \cdot \vec{v}_3 = 0$ now implies that $\frac{\sqrt{3}}{2}a + \frac{1}{2}f = 0$, or $f = -\sqrt{3}a$.

Finally, it is required that $\|\vec{v}_1\|^2 = a^2 + f^2 = a^2 + 3a^2 = 4a^2 = 1$, so that $a = \pm\frac{1}{2}$.

Let us pick $a = \frac{1}{2}$, so that $f = -\frac{\sqrt{3}}{2}$.

Summary:

$$\vec{v}_1 = \begin{bmatrix} \frac{1}{2} \\ 0 \\ -\frac{\sqrt{3}}{2} \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} \frac{\sqrt{3}}{2} \\ 0 \\ \frac{1}{2} \end{bmatrix}$$

There are other solutions; some components will have different signs.

5.1.**22** Let $W = \{\vec{x} \text{ in } \mathbb{R}^n : \vec{x} \cdot \vec{v}_i = 0 \text{ for all } i = 1, \ldots, m\}$. We are asked to show that $V^\perp = W$, that is, any $\vec{x}$ in $V^\perp$ is in W, and vice versa.

If $\vec{x}$ is in $V^\perp$, then $\vec{x} \cdot \vec{v} = 0$ for all $\vec{v}$ in V; in particular, $x \cdot \vec{v}_i = 0$ for all i (since the $\vec{v}_i$ are in V), so that $\vec{x}$ is in W.

Conversely, consider a vector $\vec{x}$ in W. To show that $\vec{x}$ is in $V^\perp$, we have to verify that $\vec{x} \cdot \vec{v} = 0$ for all $\vec{v}$ in V. Pick a particular $\vec{v}$ in V. Since the $\vec{v}_i$ span V, we can write $\vec{v} = c_1\vec{v}_1 + \cdots + c_m\vec{v}_m$, for some scalars c_i. Then $\vec{x} \cdot \vec{v} = c_1(\vec{x} \cdot \vec{v}_1) + \cdots + c_m(\vec{x} \cdot \vec{v}_m) = 0$, as claimed.

5.1.**23** We will follow the hint. Let $\vec{v}$ be a vector in V. Then $\vec{v} \cdot \vec{x} = 0$ for all $\vec{x}$ in $V^\perp$. Since $(V^\perp)^\perp$ contains all vectors $\vec{y}$ such that $\vec{y} \cdot \vec{x} = 0$, $\vec{v}$ is in $(V^\perp)^\perp$. So V is a subspace of $(V^\perp)^\perp$.

Then, by Theorem 5.1.8c, dim (V) + $\dim(V^\perp) = n$ and $\dim(V^\perp) + \dim((V^\perp)^\perp) = n$, so dim (V) + $\dim(V^\perp) = \dim(V^\perp) + \dim((V^\perp)^\perp)$ and dim $(V) = \dim((V^\perp)^\perp)$. Since V is a subspace of $(V^\perp)^\perp$, it follows that $V = (V^\perp)^\perp$, by Exercise 3.3.61.

5.1.**24** Write $T(\vec{x}) = \text{proj}_V(\vec{x})$ for simplicity.

To prove the linearity of T we will use the definition of a projection: $T(\vec{x})$ is in V, and $\vec{x} - T(\vec{x})$ is in $V^\perp$.

To show that $T(\vec{x}+\vec{y}) = T(\vec{x}) + T(\vec{y})$, note that $T(\vec{x}) + T(\vec{y})$ is in V (since V is a subspace), and $\vec{x} + \vec{y} - (T(\vec{x}) + T(\vec{y})) = (\vec{x} - T(\vec{x})) + (\vec{y} - T(\vec{y}))$ is in $V^\perp$ (since $V^\perp$ is a subspace, by Theorem 5.1.8a).

To show that $T(k\vec{x}) = kT(\vec{x})$, note that $kT(\vec{x})$ is in V (since V is a subspace), and $k\vec{x} - kT(\vec{x}) = k(\vec{x} - T(\vec{x}))$ is in $V^\perp$ (since $V^\perp$ is a subspace).

5.1.**25** a $\|k\vec{v}\|^2 = (k\vec{v}) \cdot (k\vec{v}) = k^2(\vec{v} \cdot \vec{v}) = k^2\|\vec{v}\|^2$

Now take square roots of both sides; note that $\sqrt{k^2} = |k|$, the absolute value of k (think about the case when k is negative). $\|k\vec{v}\| = |k|\|\vec{v}\|$, as claimed.

b $\|\vec{u}\| = \left\| \frac{1}{\|\vec{v}\|}\vec{v} \right\| = \frac{1}{\|\vec{v}\|}\|\vec{v}\| = 1$, as claimed.

↑

by part a

5.1.**26** The two given vectors spanning the subspace are orthogonal, but they are not unit vectors: both have length 7. To obtain an orthonormal basis $\vec{u}_1, \vec{u}_2$ of the subspace, we divide by 7:

$$\vec{u}_1 = \frac{1}{7}\begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix}, \vec{u}_2 = \frac{1}{7}\begin{bmatrix} 3 \\ -6 \\ 2 \end{bmatrix}.$$

Now we can use Theorem 5.1.5, with $\vec{x} = \begin{bmatrix} 49 \\ 49 \\ 49 \end{bmatrix}$:

$$\text{proj}_V\vec{x} = (\vec{u}_1 \cdot \vec{x})\vec{u}_1 + (\vec{u}_2 \cdot \vec{x})\vec{u}_2 = 11\begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix} - \begin{bmatrix} 3 \\ -6 \\ 2 \end{bmatrix} = \begin{bmatrix} 19 \\ 39 \\ 64 \end{bmatrix}.$$

5.1.**27** Since the two given vectors in the subspace are orthogonal, we have the orthonormal basis

$$\vec{u}_1 = \frac{1}{3}\begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \vec{u}_2 = \frac{1}{3}\begin{bmatrix} -2 \\ 2 \\ 0 \\ 1 \end{bmatrix}.$$

Now we can use Theorem 5.1.5, with $\vec{x} = 9\vec{e}_1$: $\text{proj}_V\vec{x} = (\vec{u}_1 \cdot \vec{x})\vec{u}_1 + (\vec{u}_2 \cdot \vec{x})\vec{u}_2$

$$= 2\begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \end{bmatrix} - 2\begin{bmatrix} -2 \\ 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 0 \\ 2 \\ -2 \end{bmatrix}.$$

5.1.**28** Since the three given vectors in the subspace are orthogonal, we have the orthonormal basis

$$\vec{u}_1 = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \vec{u}_2 = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \vec{u}_3 = \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}.$$

Now we can use Theorem 5.1.5, with $\vec{x} = \vec{e}_1$: $\text{proj}_V\vec{x} = (\vec{u}_1 \cdot \vec{x})\vec{u}_1 + (\vec{u}_2 \cdot \vec{x})\vec{u}_2 + (\vec{u}_3 \cdot \vec{x})\vec{u}_3 = \frac{1}{4}\begin{bmatrix} 3 \\ 1 \\ -1 \\ 1 \end{bmatrix}$.

5.1.40 $||\vec{v}_2|| = \sqrt{\vec{v}_2 \cdot \vec{v}_2} = \sqrt{a_{22}} = 3.$

5.1.41 $\theta = \arccos(\frac{\vec{v}_2 \cdot \vec{v}_3}{||\vec{v}_2|| \, ||\vec{v}_3||}) = \arccos(\frac{a_{23}}{\sqrt{a_{22}}\sqrt{a_{33}}}) = \arccos(\frac{20}{21}) \approx 0.31$ radians.

5.1.42 $||\vec{v}_1 + \vec{v}_2|| = \sqrt{(\vec{v}_1 + \vec{v}_2) \cdot (\vec{v}_1 + \vec{v}_2)} = \sqrt{a_{11} + 2a_{12} + a_{22}} = \sqrt{22}.$

5.1.43 Let $\vec{u} = \frac{\vec{v}_2}{\|\vec{v}_2\|} = \frac{\vec{v}_2}{3}$. Then, $\vec{u}$ is an orthonormal basis for span($\vec{v}_2$). Using Theorem 5.1.5, $\text{proj}_{\vec{v}_2}(\vec{v}_1) = (\vec{u}\cdot\vec{v}_1)\vec{u} = (\frac{\vec{v}_2}{3}\cdot\vec{v}_1)\frac{\vec{v}_2}{3} = \frac{1}{3}(\vec{v}_2\cdot\vec{v}_1)\frac{\vec{v}_2}{3} = \frac{1}{3}(a_{12})\frac{\vec{v}_2}{3} = \frac{5}{9}\vec{v}_2$.

5.1.44 One method to solve this is to take $\vec{v} = \vec{v}_2 - \text{proj}_{\vec{v}_3}\vec{v}_2 = \vec{v}_2 - \frac{20}{49}\vec{v}_3$.

5.1.45 Write the projection as a linear combination of $\vec{v}_2$ and $\vec{v}_3$, $c_2\vec{v}_2 + c_3\vec{v}_3$. Now you want $\vec{v}_1 - c_2\vec{v}_2 - c_3\vec{v}_3$ to be perpendicular to V, that is, perpendicular to both $\vec{v}_2$ and $\vec{v}_3$. Using dot products, this boils down to two linear equation in two unknowns, $9c_2 + 20c_3 = 5$, and $20c_2 + 49c_3 = 11$, with the solution $c_2 = \frac{25}{41}$ and $c_3 = -\frac{1}{41}$. Thus the answer is $\frac{25}{41}\vec{v}_2 - \frac{1}{41}\vec{v}_3$.

5.1.46 Write the projection as a linear combination of $\vec{v}_1$ and $\vec{v}_2$: $c_1\vec{v}_1 + c_2\vec{v}_2$. Now we want $\vec{v}_3 - c_1\vec{v}_1 + c_2\vec{v}_2$ to be perpendicular to V, that is, perpendicular to both $\vec{v}_1$ and $\vec{v}_2$. Using dot products, this boils down to two linear equations in two unknowns, $11 = 3c_1 + 5c_2$ and $20 = 5c_1 + 9c_2$, with the solution $c_1 = -\frac{1}{2}, c_2 = \frac{5}{2}$. Thus, the answer is $-\frac{1}{2}\vec{v}_1 + \frac{5}{2}\vec{v}_2$.

Section 5.2

In Exercises 1–14, we will refer to the given vectors as $\vec{v}_1, \ldots, \vec{v}_m$, where $m = 1, 2$, or 3.

5.2.1 $\vec{u}_1 = \frac{1}{\|\vec{v}_1\|}\vec{v}_1 = \frac{1}{3}\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$

5.2.2 $\vec{u}_1 = \frac{1}{\|\vec{v}_1\|}\vec{v}_1 = \frac{1}{7}\begin{bmatrix} 6 \\ 3 \\ 2 \end{bmatrix}$

$$\vec{u}_2 = \frac{\vec{v}_2^{\perp}}{\|\vec{v}_2^{\perp}\|} = \frac{\vec{v}_2 - (\vec{u}_1\cdot\vec{v}_2)\vec{u}_1}{\|\vec{v}_2 - (\vec{u}_1\cdot\vec{v}_2)\vec{u}_1\|} = \frac{1}{7}\begin{bmatrix} 2 \\ -6 \\ 3 \end{bmatrix}$$

Note that $\vec{u}_1 \cdot \vec{v}_2 = 0$.

5.2.3 $\vec{u}_1 = \frac{1}{\|\vec{v}_1\|}\vec{v}_1 = \frac{1}{5}\begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix}$

$$\vec{u}_2 = \frac{\vec{v}_2^{\perp}}{\|\vec{v}_2^{\perp}\|} = \frac{\vec{v}_2 - (\vec{u}_1\cdot\vec{v}_2)\vec{u}_1}{\|\vec{v}_2 - (\vec{u}_1\cdot\vec{v}_2)\vec{u}_1\|} = \frac{1}{5}\begin{bmatrix} 3 \\ 0 \\ -4 \end{bmatrix}$$

5.2.4 $\vec{u}_1 = \frac{1}{5}\begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix}$ and $\vec{u}_2 = \frac{1}{5}\begin{bmatrix} 3 \\ 0 \\ -4 \end{bmatrix}$ as in Exercise 3.

Since $\vec{v}_3$ is orthogonal to $\vec{u}_1$ and $\vec{u}_2$, $\vec{u}_3 = \frac{1}{\|\vec{v}_3\|}\vec{v}_3 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$.

5.2.12 $\vec{u}_1 = \frac{1}{7}\begin{bmatrix} 2 \\ 3 \\ 0 \\ 6 \end{bmatrix}$

$$\vec{u}_2 = \frac{\vec{v}_2^{\perp}}{\|\vec{v}_2^{\perp}\|} = \frac{\vec{v}_2 - (\vec{u}_1 \cdot \vec{v}_2)\vec{u}_1}{\|\vec{v}_2 - (\vec{u}_1 \cdot \vec{v}_2)\vec{u}_1\|} = \frac{1}{3}\begin{bmatrix} 0 \\ -2 \\ 2 \\ 1 \end{bmatrix}$$

In Exercises 15–28, we will use the results of Exercises 1–14 (note that Exercise k, where $k = 1, \ldots, 14$, gives the QR factorization of the matrix in Exercise $(k+14)$). We can set $Q = [\vec{u}_1 \ldots \vec{u}_m]$; the entries of R are

$$\begin{aligned} r_{11} &= \|\vec{v}_1\| & & \\ r_{22} &= \|\vec{v}_2^{\perp}\| & &= \|\vec{v}_2 - (\vec{u}_1 \cdot \vec{v}_2)\vec{u}_1\| \\ r_{33} &= \|\vec{v}_3^{\perp}\| & &= \|\vec{v}_3 - (\vec{u}_1 \cdot \vec{v}_3)\vec{u}_1 - (\vec{u}_2 \cdot \vec{v}_3)\vec{u}_2\| \\ r_{ij} & & &= \vec{u}_i \cdot \vec{v}_j, \text{ where } i < j. \end{aligned}$$

5.2.15 $Q = \frac{1}{3}\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}, R = [3]$

5.2.16 $Q = \frac{1}{7}\begin{bmatrix} 6 & 2 \\ 5 & -6 \\ 2 & 3 \end{bmatrix}, R = \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$

5.2.17 $Q = \frac{1}{5}\begin{bmatrix} 4 & 3 \\ 0 & 0 \\ 5 & -4 \end{bmatrix}, R = \begin{bmatrix} 5 & 5 \\ 0 & 35 \end{bmatrix}$

5.2.18 $Q = \frac{1}{5}\begin{bmatrix} 4 & 3 & 0 \\ 0 & 0 & -5 \\ 5 & -4 & 0 \end{bmatrix}, R = \begin{bmatrix} 5 & 5 & 0 \\ 0 & 35 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

5.2.32 A basis of the plane is $\vec{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

Now apply the Gram-Schmidt process.

$$\vec{u}_1 = \frac{1}{\|\vec{v}_1\|}\vec{v}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$\vec{u}_2 = \frac{\vec{v}_2^{\perp}}{\|\vec{v}_2^{\perp}\|} = \frac{\vec{v}_2-(\vec{u}_1\cdot\vec{v}_2)\vec{u}_1}{\|\vec{v}_2-(\vec{u}_1\cdot\vec{v}_2)\vec{u}_1\|} = \frac{1}{\sqrt{6}}\begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$$

Your solution may be different if you start with a different basis $\vec{v}_1, \vec{v}_2$ of the plane.

5.2.33 $\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$

A basis of $\ker(A)$ is $\vec{v}_1 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$.

Since $\vec{v}_1$ and $\vec{v}_2$ are orthogonal already, we obtain $\vec{u}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, $\vec{u}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$.

5.2.34 $\text{rref}(A) = \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \end{bmatrix}$

A basis of $\ker(A)$ is $\vec{v}_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}$.

We apply the Gram-Schmidt process and obtain

$$\vec{u}_1 = \frac{1}{\|\vec{v}_1\|}\vec{v}_1 = \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}$$

$$\vec{u}_2 = \frac{\vec{v}_2^{\perp}}{\|\vec{v}_2^{\perp}\|} = \frac{\vec{v}_2-(\vec{u}_1\cdot\vec{v}_2)\vec{u}_1}{\|\vec{v}_2-(\vec{u}_1\cdot\vec{v}_2)\vec{u}_1\|} = \frac{1}{\sqrt{30}}\begin{bmatrix} 2 \\ -1 \\ -4 \\ 3 \end{bmatrix}$$

5.2.35 $\text{rref}(A) = \begin{bmatrix} 1 & 0 & \frac{1}{3} \\ 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 \end{bmatrix}$

The non-redundant columns of A give us a basis of $\text{im}(A)$:

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$

Since $\vec{v}_1$ and $\vec{v}_2$ are orthogonal already, we obtain $\vec{u}_1 = \frac{1}{3}\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$, $\vec{u}_2 = \frac{1}{3}\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$.

5.2.36 Write $M = \underbrace{\frac{1}{2}\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}}_{Q_0} \underbrace{\begin{bmatrix} 2 & 3 & 5 \\ 0 & -4 & 6 \\ 0 & 0 & 7 \end{bmatrix}}_{R_0}$

This is almost the QR factorization of M: the matrix Q_0 has orthonormal columns and R_0 is upper triangular; the only problem is the entry -4 on the diagonal of R_0. Keeping in mind how matrices are multiplied, we can change all the signs in the second column of Q_0 and in the second row of R_0 to fix this problem:

$$M = \underbrace{\frac{1}{2}\begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} 2 & 3 & 5 \\ 0 & 4 & -6 \\ 0 & 0 & 7 \end{bmatrix}}_{R}$$

5.2.37 Write $M = \underbrace{\frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{bmatrix}}_{Q_0} \underbrace{\begin{bmatrix} 3 & 4 \\ 0 & 5 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}}_{R_0}$

Note that the last two columns of Q_0 and the last two rows of R_0 have no effect on the product Q_0R_0; if we drop them, we have the QR factorization of M:

$$M = \underbrace{\frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} 3 & 4 \\ 0 & 5 \end{bmatrix}}_{R}$$

5.2.38 Since $\vec{v}_1 = 2\vec{e}_3$, $\vec{v}_2 = -3\vec{e}_1$ and $\vec{v}_3 = 4\vec{e}_4$ are orthogonal, we have

$$Q = \begin{bmatrix} \frac{\vec{v}_1}{\|\vec{v}_1\|} & \frac{\vec{v}_2}{\|\vec{v}_2\|} & \frac{\vec{v}_3}{\|\vec{v}_3\|} \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } R = \begin{bmatrix} \|\vec{v}_1\| & 0 & 0 \\ 0 & \|\vec{v}_2\| & 0 \\ 0 & 0 & \|\vec{v}_3\| \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}.$$

5.2.44 No! If m exceeds n, then there is no $n \times m$ matrix Q with orthonormal columns (if the columns of a matrix are orthonormal, then they are linearly independent).

5.2.45 Yes. Let $A = [\,\vec{v}_1 \quad \cdots \quad \vec{v}_m\,]$. The idea is to perform the Gram-Schmidt process in reversed order, starting with $\vec{u}_m = \frac{1}{\|\vec{v}_m\|}\vec{v}_m$.

Then we can express $\vec{v}_j$ as a linear combination of $\vec{u}_j, \ldots, \vec{u}_m$, so that $[\,\vec{v}_1 \quad \cdots \quad \vec{v}_j \quad \cdots \quad \vec{v}_m\,] = [\,\vec{u}_1 \quad \cdots \quad \vec{u}_j \quad \cdots \quad \vec{u}_m\,]\,L$ for some *lower* triangular matrix L, with