Homework #1, Math 28

due September 12, 2012

1. Place a knight on each square of a 7-by-7 chessboard. Is it possible for each knight to simultaneously make a legal move? (A legal knight move is to move two squares horizontally and one square vertically, or two squares vertically and one square horizontally. Note: Two knights don't fit on the same square).

2. In a party with 45 people, among any set of four there is at least one person who knows each of the other three. There are three people who are not mutually acquainted with each other. Prove that the other 42 people know everyone at the party. (Assume that "knowing" is a symmetric relation, i.e., if A knows B then B knows A).

3. Show that if you have a party with at least 6 people, there are three people who are all either mutual acquaintances (each one knows the other two) or mutual strangers (each one does not know either of the other two). (Assume that "knowing" is a symmetric relation, i.e., if A knows B then B knows A).

4. Show, for all integers $n \geq 1$, that
$$1^3 + 2^3 + \ldots n^3 = \left(\frac{n(n+1)}{2}\right)^2.$$

5. Calculate
$$\frac{1}{1 \times 3} + \frac{1}{3 \times 5} + \ldots + \frac{1}{999 \times 1001}.$$

6. Show that the product of three consecutive integers is a multiple of 6.

7. The following statement is obviously false: "In any list of n integers, all of them are equal". Find the mistake in the following "induction" proof: "For $n = 1$, the statement is obviously true since the list has only one number. Now suppose the statement is true for any list of k integers. Consider an arbitrary list of $k+1$ integers: $a_1, a_2, \ldots a_k, a_{k+1}$. By the inductive hypothesis, taking the k-integer list $\{a_1, a_2, \ldots, a_k\}$, we have $a_1 = a_2 = \ldots = a_k$. Also, by taking the k-integer list $\{a_2, a_3, \ldots, a_k, a_{k+1}\}$, we have $a_2 = a_3 = \ldots = a_k = a_{k+1}$. Therefore all the integers in the list are equal to a_2 and hence all of them are equal."