Final Practice Exam Linear Algebra

1. Answer True or False on the following questions:

a) If A and B are symmetric $n \times n$ matrices then $A + B$ is also symmetric.

b) If A is an invertible matrix then $(A^T)^{-1} = (A^{-1})^T$.

c) For any scalar c, $||c\vec{v}|| = c||\vec{v}||$.

d) If $\vec{x}$ is orthogonal to every vector in a subspace W of $\mathbb{R}^n$, then $\vec{x}$ is in $W^{\perp}$.

e) The least-squares solution of $A\vec{x} = \vec{b}$ is the point in the image of A closest to $\vec{b}$.

f) If three vectors in $\mathbb{R}^3$ lie in the same plane in $\mathbb{R}^3$, then they are linearly dependent.

g) If $\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$ is an orthogonal set and if c_1, c_2, and c_3 are scalars, then $\{c_1\vec{v_1}, c_2\vec{v_2}, c_3\vec{v_3}\}$ is an orthogonal set.

h) If the columns of an $n \times n$ matrix A are orthogonal, then A is an orthogonal matrix.

i) Let W be a subspace of $\mathbb{R}^n$. If $\vec{x} \in \mathbb{R}^n$, then $\vec{x}$ - $proj_{\,W}\vec{x}$ is not zero.

j) If $\vec{x}, \vec{y} \in \mathbb{R}^n$, then $||\vec{x} + \vec{y}||^2 = ||\vec{x}||^2 + ||\vec{y}||^2$.

2. Suppose A is a 6×7 matrix that has five pivot columns.

- What is the $rank\, A$?
- What is the dimension of the row space of A?
- What is the dimension of $\ker A$?
- Is $Im(A) = \mathbb{R}^5$?

3. The matrix $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$ has two eigenvalues. Find the eigenvalues of A, and find the eigenvector associated to each eigenvalue.

4. Let $\vec{u_1} = \begin{bmatrix} -1 \\ 4 \\ -3 \end{bmatrix}$, $\vec{u_2} = \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix}$ and $\vec{u_3} = \begin{bmatrix} 3 \\ -4 \\ -7 \end{bmatrix}$. Is the set $\{\vec{u_1}, \vec{u_2}, \vec{u_3}\}$ orthogonal?

5. Let $\vec{y} = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$, and $\vec{v} = \begin{bmatrix} -4 \\ 2 \end{bmatrix}$. Let $W = Span\,\{\vec{v}\}$.

- What is the orthogonal projection of $\vec{y}$ to W?
- What is the orthogonal projection of $\vec{y}$ to the line through $\vec{v}$ and the origin?
- What is the length of the projection of $\vec{y}$ onto W?
- What is the distance from $\vec{y}$ to W.

6. Find an orthogonal basis $\{\vec{u_1}, \vec{u_2}, \vec{u_3}\}$ for the column space of the matrix

$$A = \begin{bmatrix} 3 & -5 & 1 \\ 1 & 1 & 1 \\ 3 & -7 & 8 \end{bmatrix}.$$

7. Let $A = \begin{bmatrix} 1 & -2 \\ -1 & 2 \\ 0 & 3 \\ 2 & 5 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 3 \\ 1 \\ -4 \\ 2 \end{bmatrix}$.

- Find $A^T A$.
- Find $A^T \vec{b}$.
- Find the least squares solution of $A\vec{x} = \vec{b}$.

8. Find the equation $y = mx+b$ of the least-squares line that best fits the data points $(2,3), (3,2), (5,1), (6,0)$.

9. Prove $\ker A = \ker(A^T A)$ for any $m \times n$ matrix A.

10. Find the determinant of

$$\begin{vmatrix} 4 & 0 & -7 & 3 & -5 \\ 0 & 0 & 2 & 0 & 0 \\ 7 & 3 & -6 & 4 & -8 \\ 5 & 0 & 5 & 2 & -3 \\ 0 & 0 & 9 & -1 & 2 \end{vmatrix}.$$

11. Let $A = \begin{bmatrix} .75 & .25 \\ .25 & .75 \end{bmatrix}$.

- Find the eigenvalues of A.
- Find the eigenvector associated to each eigenvalue.
- Write A as PDP^{-1}, where D is a diagonal matrix. (Note that you have to find P, D and P^{-1}).
- Show that A^k approaches $\begin{bmatrix} .5 & .5 \\ .5 & .5 \end{bmatrix}$ as $k \to \infty$.

12. Consider the polynomials $\mathbf{p_1}(t) = 1+t, \mathbf{p_2}(t) = 1-t, \mathbf{p_3}(t) = 4, \mathbf{p_4}(t) = t+t^2$, and $\mathbf{p_5}(t) = 1+2t+t^2$, and let H be the subspace of $\mathbb{P}_5$ spanned by the set $S = \{\mathbf{p_1}, \mathbf{p_2}, \mathbf{p_3}, \mathbf{p_4}, \mathbf{p_5}\}$. Find a basis for H.

BONUS: December 12 is a popular holiday in Mexico. What is this holiday?.