NAME: Enrique

SECTION:

MATH 28 MIDTERM #1

October 4, 2012

INSTRUCTIONS: This is a closed book, closed notes exam. You are not to provide or receive help from any outside source during the exam.

- You may NOT use a calculator.
- Show all of your work.

Question	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
6	20	
7	20	
8	20	
9	20	
10	20	
Total:	200	

1. Answer True or False in the following questions. Please do so on the left side.

FALSE (a) [2 points] The function $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} y \\ 1 \end{bmatrix}$ is a linear transformation.

$T\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

TRUE (b) [2 points] Any linear combination of vectors can always be written in the form $A\vec{x}$ for a suitable matrix A and a vector $\vec{x}$.

TRUE (c) [2 points] The matrix $\begin{bmatrix} -0.6 & 0.8 \\ -0.8 & -0.6 \end{bmatrix}$ represents a rotation.

$(-0.6)^2 + (-0.8)^2 = .36 + .64 = 1$ so the matrix is of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ where $a^2+b^2=1$.

FALSE (d) [2 points] The formula $AB = BA$ holds for all $n \times n$ matrices A and B.

Because matrix multiplication is not commutative.

FALSE (e) [2 points] If A is a 3×2 matrix, then the transformation $\vec{x} \mapsto A\vec{x}$ cannot be one to one.

Example: $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$.

TRUE (f) [2 points] If $A^2 = I_n$ then the matrix A must be invertible.

$(A)(A) = I_n$, so $A^{-1} = A$.

TRUE (g) [2 points] Every linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is a matrix transformation.

By Theorem

TRUE (h) [2 points] If A is any invertible $n \times n$ matrix, then $rref(A) = I_n$.

By Theorem

TRUE (i) [2 points] If $n \times n$ matrices E and F have the property that $EF = I$, then E and F commute, i.e., $EF = FE$.

$F = E^{-1}$ so $FE = E^{-1}E = I = EF$

TRUE (j) [2 points] An example of a linear combination of vectors $\vec{v_1}$ and $\vec{v_2}$ is the vector $\frac{1}{2}\vec{v_1}$.

$\frac{1}{2}\vec{v_1} = \frac{1}{2}\vec{v_1} + 0\,\vec{v_2}$

2. [20 points] At the beginning of a semester, 55 students have signed up for Linear Algebra; the course is offered in two sections that are taught at different times. Because of scheduling conflicts and personal preferences, 20% of the students in section A switch to section B in the first few weeks of class, while 30% of the students in section B switch to section A, resulting in a net loss of 4 students for section B. Given that no students dropped Linear Algebra or joined the course; how large were the two sections at the beginning of the semester?

Let a = # students in A at the beginning of the semester
b = # students in B at the beginning of the semester.

$$a + b = 55$$

Then $$b + 0.2a - 0.3b = b - 4$$

so

$$a + b = 55$$
$$0.2a - 0.3b = -4$$

$$\left[\begin{array}{cc|c} 1 & 1 & 55 \\ 0.2 & -0.3 & -4 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 55 \\ 0 & -0.5 & -15 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 55 \\ 0 & 1 & 30 \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{ccc} 1 & 0 & 25 \\ 0 & 1 & 30 \end{array}\right]$$

Therefore $a = 25$ and $b = 30$

9. Let $f_1, f_2, \ldots$ be the Fibonacci sequence, i.e., $f_1 = 1, f_2 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 3$.

(a) [10 points] Prove that $f_1^2 + f_2^2 + \ldots + f_n^2 = f_n f_{n+1}$.

For $n=1$ and $n=2$ we have

✓ $1 = f_1^2 = f_1 f_2$ and $2 = f_1^2 + f_2^2 = f_2 f_3 = (1)(2)$ ✓

Suppose $f_1^2 + \ldots + f_k^2 = f_k f_{k+1}$ when $k \geq 2$.

Now we want to show $f_1^2 + f_2^2 + \ldots + f_k^2 + f_{k+1}^2 = f_{k+1} f_{k+2}$

$$f_1^2 + f_2^2 + \ldots + f_k^2 + f_{k+1}^2 = f_k f_{k+1} + f_{k+1}^2$$

$$= f_{k+1}(f_k + f_{k+1})$$

$$= f_{k+1}(f_{k+2})$$ ✓

Hence the proof is complete (by induction).

(b) [10 points] Prove that $\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}^n = \begin{pmatrix} f_{n+1} & f_n \\ f_n & f_{n-1}\end{pmatrix}$ for $n \geq 2$.

For $n=2$, $\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}^2 = \begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix} = \begin{pmatrix}2 & 1\\ 1 & 1\end{pmatrix} = \begin{pmatrix}f_3 & f_2\\ f_2 & f_1\end{pmatrix}$ ✓

Assume that $\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}^k = \begin{pmatrix}f_{k+1} & f_k\\ f_k & f_{k-1}\end{pmatrix}$ for $k \geq 2$.

Now

$$\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}^{k+1} = \begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}^k \begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}$$

$$= \begin{pmatrix}f_{k+1} & f_k\\ f_k & f_{k-1}\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix} = \begin{pmatrix}f_{k+1}+f_k & f_{k+1}+0\\ f_k+f_{k-1} & f_k+0\end{pmatrix}$$

$$= \begin{pmatrix}f_{k+2} & f_{k+1}\\ f_{k+1} & f_k\end{pmatrix}$$

so by induction we've shown that

$\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}^n = \begin{pmatrix}f_{n+1} & f_n\\ f_n & f_{n-1}\end{pmatrix}$ for $n \geq 2$.

4. Consider the system of equations

$$x_1 + hx_2 = 4$$
$$3x_1 + 9x_2 = k,$$

for real numbers h and k.

(a) [5 points] Choose h and k in such a way that the system of linear equations has no solution.

$$\left[\begin{array}{cc|c} 1 & h & 4 \\ 3 & 9 & k \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & h & 4 \\ 0 & 9-3h & k-12 \end{array}\right]$$

For there to be no solutions we need $9-3h=0$ and $k-12 \neq 0$.

So let $h=3$ and $k=13$. (Note: k can be anything besides 12).

(b) [5 points] Choose h and k in such a way that the system of linear equations has a unique solution.

If $h \neq 3$, then the rank of the matrix is 2 and hence there is a solution.

Any $h \neq 3$ and any k suffices.

Let $h=0$ and $k=12$ (We have to choose one).

(c) [2 points] What is the unique solution in this case?

If $h=0$ and $k=12$, the solution is $x_2=0$ and $x_1=4$.

(d) [5 points] Choose h and k in such a way that the system of linear equations has many solutions.

We need $h=3$ (so that the rank is 1)

and we need $k=12$ (o.w there's no solution).

So $\boxed{\begin{matrix} h=3 \\ k=12 \end{matrix}}$

(e) [3 points] Describe the solution set in this case?

$$\left[\begin{array}{cc|c} 1 & 3 & 4 \\ 0 & 0 & 0 \end{array}\right]$$

so $x_1 + 3x_2 = 4$

so $x_1 = 4 - 3x_2$

so $\vec{x} = \begin{bmatrix} 4 - 3x_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix} - \begin{bmatrix} 3 \\ 1 \end{bmatrix} x_2$

are the solutions to this system of equations.

6. [20 points] Leonard has to solve the system of linear equations

$$A\vec{x} = \begin{bmatrix} 2 \\ 7 \\ -1 \\ 0 \end{bmatrix}, \text{ where } A = \begin{pmatrix} 5 & 8 & 6 & 2 \\ 3 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 2 & 4 & 3 & 1 \end{pmatrix}.$$

His roommate Sheldon, after glancing over Leonard's shoulder, informs him (correctly) that

$$A^{-1} = \begin{pmatrix} 1 & 0 & 0 & -2 \\ 3 & -1 & 0 & -6 \\ 3 & -1 & -1 & -6 \\ -23 & 7 & 3 & 47 \end{pmatrix}.$$

With that information, Leonard quickly solved the system. What solution did Leonard find (assuming, of course, that Leonard found the correct solution)?

$$\vec{x} = A^{-1}\begin{bmatrix} 2 \\ 7 \\ -1 \\ 0 \end{bmatrix} = \begin{pmatrix} 1 & 0 & 0 & -2 \\ 3 & -1 & 0 & -6 \\ 3 & -1 & -1 & -6 \\ -23 & 7 & 3 & 47 \end{pmatrix}\begin{bmatrix} 2 \\ 7 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2+0+0-2(0) \\ 6-7+0+0 \\ 6-7+1+0 \\ -46+49-3+0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \end{bmatrix}.$$

Indeed

$$\begin{pmatrix} 5 & 8 & 6 & 2 \\ 3 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 2 & 4 & 3 & 1 \end{pmatrix}\begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 10-8 \\ 6+1 \\ 0+(-1) \\ 4-4 \end{bmatrix} = \begin{bmatrix} 2 \\ 7 \\ -1 \\ 0 \end{bmatrix}.$$

5. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that satisfies $T(\vec{u}) = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$ and $T(\vec{v}) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, where $\vec{u} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$.

(a) [5 points] Find $T(\vec{e_1})$ where $\vec{e_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

We want to write $\vec{e}_1$ in terms of $\vec{u}$ and $\vec{v}$.

$$\left[\begin{array}{cc|c} 5 & 4 & 1 \\ 5 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 5 & 4 & 1 \\ 0 & -2 & -1 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 4/5 & 1/5 \\ 0 & 1 & 1/2 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & -1/5 \\ 0 & 1 & 1/2 \end{array}\right]$$

so $-\frac{1}{5}\begin{bmatrix} 5 \\ 5 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Hence

$$T(\vec{e}_1) = T\left(-\frac{1}{5}\begin{bmatrix} 5 \\ 5 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 4 \\ 2 \end{bmatrix}\right) = -\frac{1}{5}T\left(\begin{bmatrix} 5 \\ 5 \end{bmatrix}\right) + \frac{1}{2}T\left(\begin{bmatrix} 4 \\ 2 \end{bmatrix}\right)$$

$$= -\frac{1}{5}\begin{bmatrix} 2 \\ 6 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -2/5 + 1/2 \\ -\frac{6}{5} + \frac{3}{2} \end{bmatrix} = \begin{bmatrix} 1/10 \\ 3/10 \end{bmatrix}$$

(b) [5 points] Find $T(\vec{e_2})$ where $\vec{e_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Similarly

$$\begin{bmatrix} 5 & 4 & 0 \\ 5 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 4/5 & 0 \\ 0 & -2 & 1 \end{bmatrix} \sim \left[\begin{array}{cc|c} 1 & 4/5 & 0 \\ 0 & 1 & -1/2 \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{cc|c} 1 & 0 & 2/5 \\ 0 & 1 & -1/2 \end{array}\right]$$

$$T(\vec{e}_2) = \frac{2}{5}T\left(\begin{bmatrix} 5 \\ 5 \end{bmatrix}\right) - \frac{1}{2}T\left(\begin{bmatrix} 4 \\ 2 \end{bmatrix}\right) = \frac{2}{5}\begin{bmatrix} 2 \\ 6 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 4/5 - 1/2 \\ \frac{12}{5} - 3/2 \end{bmatrix} = \begin{bmatrix} 3/10 \\ 9/10 \end{bmatrix}$$

(c) [5 points] Show that T is an orthogonal projection onto a line L.

So $T(\vec{x}) = \begin{pmatrix} 1/10 & 3/10 \\ 3/10 & 9/10 \end{pmatrix} \vec{x} = A\vec{x}$

Let $u_1 = \sqrt{1/10}$, $u_2 = \sqrt{9/10}$, then $u_1^2 = 1/10$, $u_2^2 = 9/10$, $u_1 u_2 = 3/10$

and $u_1^2 + u_2^2 = 1$.

Since A is of the form $\begin{pmatrix} u_1^2 & u_1 u_2 \\ u_1 u_2 & u_2^2 \end{pmatrix}$ with $u_1^2 + u_2^2 = 1$, T is an orthogonal projection onto the line associated to the vector $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$.

(d) [5 points] What is the equation of the line L?

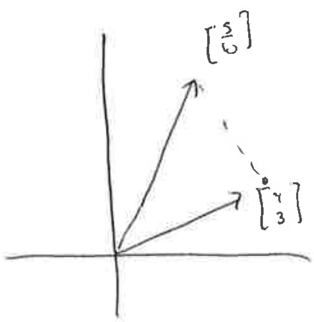
L is the line that goes through the origin and $\left(1/\sqrt{10}, 3/\sqrt{10}\right)$. The slope is hence $\frac{3/\sqrt{10}}{1/\sqrt{10}} = 3$

Since the slope is 3 and it goes through the origin, the equation is $y = 3x$.

10. Let $\vec{a} = \begin{bmatrix} 5 \\ 10 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$.

(a) [5 points] Find the projection $\text{proj}_{\vec{b}}\vec{a}$ of $\vec{a}$ onto $\vec{b}$.

$$\text{proj}_{\vec{b}}\vec{a} = \frac{\vec{a}\cdot\vec{b}}{\|\vec{b}\|^2}\vec{b} = \frac{20+30}{\left(\sqrt{3^2+4^2}\right)^2}\begin{bmatrix} 4 \\ 3 \end{bmatrix} = \frac{50}{25}\begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 8 \\ 6 \end{bmatrix}$$



(b) [5 points] Find the distance between the vectors $\vec{a}$ and $\text{proj}_{\vec{b}}\vec{a}$.

$$\left\| \begin{bmatrix} 5 \\ 10 \end{bmatrix} - \begin{bmatrix} 8 \\ 6 \end{bmatrix} \right\| = \left\| \begin{bmatrix} -3 \\ 4 \end{bmatrix} \right\| = \sqrt{3^2+4^2} = 5$$

(c) [10 points] Consider the triangle formed by the points $(0,0), (5,10)$ and $(4,3)$. Using the area formula for triangles (base times height divided by 2) and (b), find the area of the triangle.

The length of the base is $\|\vec{b}\| = \sqrt{3^2+4^2} = 5$

and the length of the height is 5 (from (b))

so the area

is

$$\frac{(5)(5)}{2} = \boxed{\frac{25}{2}}$$

7. Let $\vec{x} = \begin{bmatrix} 5 \\ 3 \\ -9 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$.

(a) [5 points] Find a diagonal matrix A such that $A\vec{x} = \vec{y}$. (Note: A diagonal matrix is a matrix whose only nonzero entries are in the diagonal).

$$\underbrace{\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ -9 \end{bmatrix}} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 5a \\ 3b \\ -9c \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

so $a = 2/5$, $b = 0$, $c = -1/9$,

$$A = \begin{bmatrix} 2/5 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1/9 \end{bmatrix}$$

(b) [5 points] Find a matrix A of rank 1 such that $A\vec{x} = \vec{y}$.

$$\begin{bmatrix} a & b & c \\ 0 & 0 & 0 \\ ka & kb & kc \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ -9 \end{bmatrix} = \begin{bmatrix} 5a + 3b - 9c \\ 0 \\ 5ka + 3kb - 9kc \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

so $k = 1/2$

We want $5a + 3b - 9c = 2$

Let $a = 2/5$, $b = 0$, $c = 0$.

$$A = \begin{bmatrix} 2/5 & 0 & 0 \\ 0 & 0 & 0 \\ 1/5 & 0 & 0 \end{bmatrix}.$$

Indeed $\begin{bmatrix} 2/5 & 0 & 0 \\ 0 & 0 & 0 \\ 1/5 & 0 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ -9 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$.

(c) [5 points] Find a matrix A of rank 2 such that $A\vec{x} = \vec{y}$.

$$\begin{bmatrix} a & b & c \\ 0 & 0 & 0 \\ d & e & f \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ -9 \end{bmatrix} = \begin{bmatrix} 5a+3b-9c \\ 0 \\ 5d+3e-9f \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

let $a = 2/5$, $b=0$, $c=0$
$d=0$, $e=1/3$, $f=0$

$$A = \begin{bmatrix} 2/5 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1/3 & 0 \end{bmatrix}$$

Indeed $\begin{bmatrix} 2/5 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1/3 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ -9 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ ✓

(d) [5 points] Find a matrix A with all nonzero entries such that $A\vec{x} = \vec{y}$.

$5a_1 + 3a_2 - 9a_3 = 2$
$5a_4 + 3a_5 - 9a_6 = 0$
$5a_7 + 3a_8 - 9a_9 = 1$

$a_1 = 4$, $a_2 = 3$, $a_3 = 3$
$20 + 9 - 27 = 2$ ✓
$a_4 = 9$, $a_5 = 3$, $a_6 = 6$

$$A = \begin{pmatrix} 4 & 3 & 3 \\ 9 & 3 & 6 \\ 2 & 3/2 & 3/2 \end{pmatrix}$$

Indeed $\begin{pmatrix} 4 & 3 & 3 \\ 9 & 3 & 6 \\ 2 & 3/2 & 3/2 \end{pmatrix} \begin{bmatrix} 5 \\ 3 \\ -9 \end{bmatrix} = \begin{bmatrix} 20+9-27 \\ 45+9-54 \\ 10+9/2-\frac{27}{2} \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ ✓

BONUS (10 points total)

1. On what date did the Mexican Independence war start?

Sept. 16, 1810

2. Where (name of the city) did the war start?

Dolores

3. Who led the rebellion?

Miguel Hidalgo y Costilla

4. What symbol (flag) did the rebellion use?

The virgin of Guadalupe

5. Name one Mexican Independence hero besides the leader of the rebellion.

Ignacio Allende,
José María y Morelos,
Vicente Guerrero,
Francisco Javier Mina