Worksheet #2, Determinants, Math 28

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For an $n \times n$ matrix A, let A_{ij} be the matrix obtained by omitting the i-th row and j-th column of A. For example if

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix},$$

then

$$A_{22} = \begin{pmatrix} 1 & 3 \\ 7 & 9 \end{pmatrix}, A_{13} = \begin{pmatrix} 4 & 5 \\ 7 & 8 \end{pmatrix}.$$

The determinant of an $n \times n$ matrix A will be denoted by $|A|$ or $\det A$. The definition of the determinant is the following: For any column j

$$det(A) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij}, \tag{1}$$

and for any row i

$$det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij}, \tag{2}$$

Note that in formula (1) you pick a column that you will go through, that is why the j stays constant throughout. Analogously, for formula (2) you pick a row i and you go through it.

Example: If

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 3 & 0 & 2 \end{pmatrix},$$

then we will pick the second column (since it has more zeros) and use the formula with the second column:

$$\begin{aligned}\det A &= (-1)^{1+2}(2)\det A_{12} + (-1)^{2+2}(0)\det A_{22} + (-1)^{3+2}(0)\det A_{32} \\ &= -2\left|\begin{pmatrix}1 & 1\\ 3 & 2\end{pmatrix}\right| + 0 + 0 = -2(2-3) = 2.\end{aligned}$$

It turns out that A is invertible if and only if $\det A \neq 0$. If the determinant is 0 we say the matrix is singular. Another nice fact about determinants is that the determinant of the product of two $n \times n$ matrices A, B is the same as the product of the determinants of A and B, i.e., $\det(AB) = (\det A)(\det B)$.

Exercises: For the exercises assume A is an $n \times n$ matrix.

1. Show that the formula (**??**) is consistent with the formula you know for the determinant of a 2×2 matrix.
2. Find the determinant of
$$\begin{pmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ 7 & 8 & 9\end{pmatrix}.$$
3. Find the determinant of
$$\begin{pmatrix}1 & 0 & 1 & 2\\ 9 & 1 & 3 & 0\\ 9 & 2 & 2 & 0\\ 5 & 0 & 0 & 3\end{pmatrix}.$$
4. Prove that $\det A = \det A^T$.
5. Let B be the matrix where the first row of A is multiplied by a constant k and the rest of A stays the same. What is $\det B$?
6. Let $B = kA$ for a scalar k, what is $\det B$.
7. Let B be a matrix where two rows of A are swapped. Prove that $\det B = -\det A$.
8. If A is invertible what is $\det A^{-1}$?
9. If B is obtained from A by adding a multiple of a row of A to another row, prove that $\det B = \det A$.

10. Think about how you can use Exercises 9, 7 and 5 together with rref to figure out another way of finding the determinant of a matrix.