MATH 77: ALGEBRAIC NUMBER THEORY
HOMEWORK 1

(1) Prove Fermat's Last Theorem in the case $n = 4$, i.e., prove that there are no positive integers x, y, z satisfying $x^4 + y^4 = z^4$.

(2) Let X and Y be ideals of the ring R. Show that $X + Y$ and XY are ideals too.

(3) Let K be a field and p and q be nonzero polynomials over K such that $\deg p \geq \deg q$. Show that the Euclidean algorithm for p and q yields the greatest common divisor of p and q. To show this, let n be a posiive integer and let $r_1, r_2, \ldots, r_n$, and $s_1, s_2, \ldots s_{n+1}$ be polynomials over K satisfying the following :

$$\begin{aligned} p =& q s_1 + r_1 && \text{where } \deg r_1 < \deg q \leq \deg p \\ q =& r_1 s_2 + r_2 && \deg r_2 < \deg r_1 \\ &\vdots && \vdots \\ r_{n-2} =& r_{n-1} s_n + r_n && \deg r_n < \deg r_{n-1} \\ r_{n-1} =& r_n s_{n+1}. \end{aligned}$$

Show that r_n is the greatest common divisor of p and q.

(4) Let K be a field. Prove that any nonzero polynomial $f \in K[t]$ can be factored into finitely many irreducible factors and that these are unique up to the order in which they are multiplied and constant factors.

(5) Let $P(x) = c_n x^n + c_{n-1} x^{n-1} + \cdots + c_0$ be a polynomial with integer coefficients. Suppose that r is a rational number such that $P(r) = 0$. Show that the n numbers

$$c_n r,\ c_n r^2 + c_{n-1} r,\ c_n r^3 + c_{n-1} r^2 + c_{n-2} r,$$
$$\ldots,\ c_n r^n + c_{n-1} r^{n-1} + \cdots + c_1 r$$

are integers.