MATH 77: ALGEBRAIC NUMBER THEORY
HOMEWORK 2

(1) Let L be a field extension of K, and $\alpha \in L$. Prove that the minimal polynomial of α over K is irreducible over K.

(2) Let $\{x_1, x_2, \ldots, x_n\}$ and $\{y_1, y_2, \ldots y_n\}$ be a bases of a free abelian group G. Let A and be B be the matrices formed of $a_{i\,j}$ and $b_{i\,j}$ respective, where $a_{i\,j}$ and $b_{i\,j}$ satisfy

$$y_i = \sum_j a_{i\,j} x_j, \qquad x_i = \sum_j b_{i\,j} y_j.$$

Show that $AB = I_n$.

(3) Which of the following polynomials are irreducible over $\mathbb{Z}$?
a) $x^2 + 3$
b) $x^2 - 169$
c) $x^3 + x^2 + x + 1$
d) $x^3 + 2x^2 + 3x + 4$.

(4) Find the minimum polynomial over $\mathbb{Q}$ of
a) $\frac{1+i}{\sqrt{2}}$
b) $i + \sqrt{2}$
c) $e^{2\pi i/3} + 2$.

(5) Express the following polynomials in terms of elementary symmetric polynomials, where this is possible.
a) $t_1^2 + t_2^2 + t_3^2$
b) $t_1^3 + t_2^3$
c) $t_1 t_2^2 + t_2 t_3^2 + t_3 t_1^2$
d) $t_1 + t_2^2 + t_3^3$.

(6) Let $\mathbb{Z}$ be a $\mathbb{Z}$-module with the obvious action. Find all submodules.

(7) Let S be the set of all ordered triples (p, q, r) of prime numbers for which at least one rational number x satisfies $px^2 + qx + r = 0$. Which primes appear in seven or more elements of S? (Hint: Part of the problem can be reduced by using problem 5 from the previous homework).

BONUS Is there a finite abelian group G such that the product of the orders of all its elements is 2^{2009}?