MATH 77: ALGEBRAIC NUMBER THEORY
HOMEWORK 3

(1) Let $\mathbb{A}$ be the field of algebraic numbers. Show that $[\mathbb{A} : \mathbb{Q}] = \infty$.

(2) Let $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ and $\{\beta_1, \beta_2, \ldots, \beta_n\}$ be bases of $\mathbb{K} = \mathbb{Q}(\theta)$, and let c_{ik} be rationals such that

$$\beta_k = \sum_{i=1}^{n} c_{ik}\alpha_i.$$

Show that

$$\Delta[\beta_1, \ldots, \beta_n] = (\det(c_i k))^2 \Delta[\alpha_1, \ldots, \alpha_n].$$

(3) Let A be a Vandermonde matrix, i.e.,

$$A = \begin{pmatrix} 1 & t_1 & t_1^2 & \ldots & t_1^{n-1} \\ 1 & t_2 & t_2^2 & \ldots & t_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & t_n & t_n^2 & \ldots & t_n^{n-1} \end{pmatrix}.$$

Show that

$$\det(A) = \prod_{1 \leq i < j \leq n} (t_i - t_j).$$

(4) Which of the following complex numbers are algebraic? Which are algebraic integers?
  (a) $\frac{355}{113}$.
  (b) $e^{2\pi i/23}$.
  (c) $e^{\pi i/23}$.
  (d) $\sqrt{17} + \sqrt{19}$.
  (e) $\frac{1+\sqrt{17}}{2\sqrt{-19}}$.
  (f) $\sqrt{1+\sqrt{2}} + \sqrt{1-\sqrt{2}}$.

(5) Express $\mathbb{Q}(\sqrt{3}, \sqrt[3]{5})$ in the form $\mathbb{Q}(\theta)$.

(6) Let $\mathbb{K} = \mathbb{Q}(\sqrt[4]{2})$. Find all monomorphisms $\sigma : \mathbb{K} \to \mathbb{C}$ and the minimal polynoials (over $\mathbb{Q}$) and field polynomials (over $\mathbb{K}$) of
  (i) $\sqrt[4]{2}$,
  (ii) $\sqrt{2}$,
  (iii) 2,
  (iv) $\sqrt{2} + 1$.