MATH 77: ALGEBRAIC NUMBER THEORY
HOMEWORK 4

(1) Show that there are no nonzero algebraic integers of the form
$$\frac{1}{3}\left(a+b\theta+c\theta^2\right),$$
where $a, b, c \in \mathbb{Z}$, $0 \leq a, b, c \leq 2$ and $\theta = \sqrt[3]{5}$.
(Hint: $N(a+b\theta+c\theta^2) = a^3+5b^3+25c^3-15abc$).

(2) Show that there are no nonzero algebraic integers of the form
$$\frac{1}{3}\left(a+bt+cu\right),$$
where $a, b, c \in \mathbb{Z}$, $0 \leq a, b, c \leq 2$, $t = \sqrt[3]{175}$ and $u = \sqrt[3]{175}$.

(3) Compute integral bases and discriminants of
(a) $\mathbb{Q}(\sqrt{2}, \sqrt{3})$.
(b) $\mathbb{Q}(\sqrt[3]{2})$.

(4) Show that isomorphic number fields have the same discriminant.

(5) Let $K = \mathbb{Q}(\theta)$ where $\theta \in \mathfrak{O}_K$. Let d be the discriminant of K. Among the elements
$$\frac{1}{d}(a_0 + \ldots a_i\theta^i),$$
where $a_i \in \mathbb{Z}$ and $a_i \neq 0$, pick one with minimal value of $|a_i|$ and call it x_i. Do this for $i = 1, \ldots, n = [K : \mathbb{Q}]$. Show that $\{x_1, \ldots, x_n\}$ is a integral basis.

(6) If $\alpha_1, \ldots, \alpha_n$ are $\mathbb{Q}$-linearly independent algebraic integers in $\mathbb{Q}(\theta)$, and if
$$\Delta[\alpha_1, \ldots, \alpha_n] = d$$
where d is the discriminant of $\mathbb{Q}(\theta)$, show that $\{\alpha_1, \ldots, \alpha_n\}$ is an integral basis.