On the counting function for the generalized Niven numbers

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1 Introduction

Let $q \geq 2$ be a fixed integer and let f be an arbitrary complex-valued function defined on the set of nonnegative integers. We say that f is *completely q-additive* if

$$f(aq^j + b) = f(a) + f(b)$$

for all nonnegative integers a, b, j satisfying $b < q^j$. Given a nonnegative integer n, there exists a unique sequence $a_0(n), a_1(n), a_2(n), \ldots, \in$

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$\{0, 1, \ldots, q-1\}$ so that

$$n = \sum_{j=0}^{\infty} a_j(n) q^j. \tag{1}$$

The right hand side of expression (1) will be called the *base q expansion* of n. Using the base q expansion (1), we find that the function f is completely q-additive if and only if $f(0) = 0$ and

$$f(n) = \sum_{j=1}^{\infty} f(a_j(n)).$$

It follows that a completely q-additive function is completely determined by its values on the set $\{0, 1, \ldots, q-1\}$.

The prototypical example of a completely q-additive function is the base q sum of digits function s_q, which is defined by

$$s_q(n) = \sum_{j=0}^{\infty} a_j(n).$$

A *q-Niven number* is a nonnegative integer n that is divisible by $s_q(n)$. The question of the distribution of the q-Niven numbers can be answered by studying the counting function

$$N_q(x) = \#\{0 \le n < x \ : \ s_q(n) | n\},$$

a task which has been undertaken by several authors (see, for example, the papers of Cooper and Kennedy [1, 2, 3, 4] or De Koninck and Doyon [5]). The best known result is the asymptotic formula

$$N_q(x) = (c_q + o(1)) \frac{x}{\log x}, \text{ where } c_q = \frac{2 \log q}{(q-1)^2} \sum_{j=1}^{q-1} (j, q-1),$$

which was proven only recently by De Koninck, Doyon and Kátai [6], and independently by Mauduit, Pomerance and Sárközy [8].

Given an arbitrary non-zero, integer-valued, completely q additive function f, we define an *f-Niven number* to be a nonnegative integer n that is divisible by $f(n)$. In the final section of [6] it is suggested that the techniques used therein could be applied to derive an asymptotic expression for the counting function of the f-Niven numbers,

$$N_f(x) = \#\{0 \le n < x \mid f(n)|n\}.$$

It is the goal of this paper to show that, under an additional mild restriction on f, this is indeed the case. Our main result is the following.

Theorem 1. *Let f be an arbitrary non-zero, integer-valued, completely q-additive function and set*

$$m = \frac{1}{q}\sum_{j=0}^{q-1} f(j) \ , \ \sigma^2 = \frac{1}{q}\sum_{j=0}^{q-1} f(j)^2 - m^2,$$

$F = (f(1), f(2), \ldots, f(q-1))$ *and*

$$d = \gcd\{rf(s) - sf(r) \mid r, s \in \{1, 2, \ldots, q-1\}\}.$$

Assume $(F, q-1) = 1$.

(i) If $m \neq 0$ then for any $\epsilon \in (0, 1/2)$

$$N_f(x) = c_f \frac{x}{\log x} + O\left(\frac{x}{(\log x)^{\frac{3}{2}-\epsilon}}\right)$$

where

$$c_f = \frac{\log q}{|m|}\left(\frac{1}{q-1}\sum_{j=1}^{q-1}(j, q-1, d)\right).$$

The implied constant depends only on f and ϵ.

(ii) If $m = 0$ then

$$N_f(x) = c_f \frac{x \log\log x}{(\log x)^{\frac{1}{2}}} + O\left(\frac{x}{(\log x)^{\frac{1}{2}}}\right)$$

where

$$c_f = \left(\frac{\log q}{2\pi\sigma^2}\right)^{\frac{1}{2}}\left(\frac{1}{q-1}\sum_{j=1}^{q-1}(j, q-1, d)\right).$$

The implied constant depends only on f.

Most of the proof of this theorem is a straightforward generalization of the methods used in [6]. It is the intent of this paper, then, to indicate where significant and perhaps non-obvious modifications to the original work must be made. The first notable change is the introduction and use of the quantities d and F. Because $d = 0$ and $F = 1$ when $f = s_q$ these quantities play no role in the earlier work.

Finally, in order to shorten the presentation, we have attempted to unify the $m \neq 0$ and $m = 0$ cases for as long as possible. The result is a slightly different approach in the $m \neq 0$ case than appears in [6]. The final steps when $m = 0$ are entirely new.

2 Notation

We denote by $\mathbb{Z}, \mathbb{Z}^+, \mathbb{Z}_0^+, \mathbb{R}$ and $\mathbb{C}$ the sets of integers, positive integers, non-negative integers, real numbers and complex numbers, respectively. For the remainder of this paper, we fix an integer $q \geq 2$ and a non-zero completely q additive function $f : \mathbb{Z}_0^+ \to \mathbb{Z}$. The variable x will always be assumed real and positive and n will always be an integer. Following [6] we set

$$\begin{aligned} A(x|k,l,t) &= \#\{0 \leq n < x \,|\, n \equiv l \pmod{k} \text{ and } f(n) = t\}, \\ a(x|t) &= \#\{0 \leq n < x \,|\, f(n) = t\}. \end{aligned}$$

As usual, we use the notation $F(x) = O(G(x))$ (or $F(x) \ll G(x)$) to mean that there is a constant C so that for all sufficiently large x, $|F(x)| \leq CG(x)$. The constant C and the size of x are allowed to depend on f (and hence on q), but on no other quantity unless specified.

3 Preliminary Lemmas

For real y we define $||y||$ to be the distance from y to the nearest integer.

Lemma 1. *Let $M = \max_{1 \leq j \leq q-1} |f(j)|$. Let $s, k \in \mathbb{Z}^+$ with $(s,k) = 1$ and suppose that there is a pair $j_1, j_2 \in \{0, 1, \ldots, q-1\}$ so that $k \nmid j_1 f(j_2) - j_2 f(j_1)$. Then for any $\xi \in \mathbb{R}$*

$$\max_{0 \leq j \leq q-1} \left|\left| f(j)\xi + \frac{js}{k} \right|\right| \geq \frac{1}{2Mk}. \tag{2}$$

Proof. We first claim that the maximum in question is at least positive. To see this we argue by contradiction and assume that $||f(j)\xi + js/k|| = 0$ for $0 \le j \le q-1$. Then $f(j)\xi + js/k = n_j \in \mathbb{Z}$ for each j. Solving for ξ and equating the resulting expressions, we find that if $f(j_1), f(j_2) \ne 0$ then

$$k(n_{j_1} f(j_2) - n_{j_2} f(j_1)) = s(j_1 f(j_2) - j_2 f(j_1)). \tag{3}$$

Since $(s,k) = 1$, this implies that $k | j_1 f(j_2) - j_2 f(j_1)$, which is impossible. Since this same condition is trivially verified if $f(j_1) = 0$ or $f(j_2) = 0$, we have a contradiction in any case.

To prove the lemma we again argue by contradiction. Assume the contrary. Then

$$\begin{aligned} \frac{1}{k} &\le \left|\left|\frac{(j_1 f(j_2) - j_2 f(j_1))\, s}{k}\right|\right| \\ &= \left|\left|\left(f(j_1)f(j_2)\xi + \frac{j_1 f(j_2) s}{k}\right) - \left(f(j_1)f(j_2)\xi + \frac{j_2 f(j_1) s}{k}\right)\right|\right| \\ &\le \left|\left|f(j_2)\left(f(j_1)\xi + \frac{j_1 s}{k}\right)\right|\right| + \left|\left|f(j_1)\left(f(j_2)\xi + \frac{j_2 s}{k}\right)\right|\right|. \end{aligned}$$

By our assumption,

$$\begin{aligned} |f(j_1)| &\le M \\ &< \frac{1}{2k \max_{0\le j\le q-1} ||f(j)\xi + js/k||} \\ &\le \frac{1}{2\,||f(j_2)\xi + j_2 s/k||} \end{aligned}$$

and similarly

$$|f(j_2)| < \frac{1}{2\,||f(j_1)\xi + j_1 s/k||}.$$

Therefore, by properties of $||\cdot||$,

$$\begin{aligned}
\frac{1}{k} &\leq |f(j_2)| \cdot \left|\left| f(j_1)\xi + \frac{j_1 s}{k} \right|\right| + |f(j_1)| \cdot \left|\left| f(j_2)\xi + \frac{j_2 s}{k} \right|\right| \\
&\leq 2M \max_{0\leq j\leq q-1} \left|\left| f(j)\xi + \frac{js}{k} \right|\right| \\
&< 2M \frac{1}{2Mk} \\
&= \frac{1}{k}
\end{aligned}$$

which is a contradiction. □

It is proven in [7] that if $z_1, z_2, \ldots, z_{q-1} \in \mathbb{C}$ satisfy $|z_j| \leq 1$ for $j = 1, 2, \ldots, q-1$. then

$$\left| \frac{1}{q} \left(1 + \sum_{j=1}^{q-1} z_j \right) \right| \leq 1 - \frac{1}{2q} \max_{1\leq j\leq q-1} (1 - \text{Re } z_j). \tag{4}$$

Since for real y we have $||y||^2 \ll 1 - \cos(2\pi y)$ and $1 + y \leq e^y$, the next lemma is an immediate consequence.

Lemma 2. *Let* $e(y) = e^{2\pi i y}$. *There is a positive constant* $c_1 = c_1(f)$ *so that for any* $\xi, r \in \mathbb{R}$

$$\left| \frac{1}{q} \sum_{j=0}^{q-1} e\left(f(j)\xi + rj \right) \right| \leq exp \left(-c_1 \max_{1\leq j\leq q-1} ||f(j)\xi + rj||^2 \right). \tag{5}$$

4 The distribution of the values of f

4.1 Argument restricted to a congruence class

Let $k \in \mathbb{Z}^+$, $l \in \mathbb{Z}_0^+$ and $t \in \mathbb{Z}$. Our first goal in this section is to relate $A(x|k,l,t)$ to the function $a(x|t)$ through a series of reductions on the modulus k. The first three reductions are proven exactly as in [6], substituting our Lemma 2 for their Lemma 3. We state them for the convenience of the reader.

Reduction 1. Write $k = k_1k_2$ where k_1 is the largest divisor of k so that $(k_1, q) = 1$. Then the primes dividing k_2 also divide q and we let h be the smallest positive integer so that k_2 divides q^h. Since k divides k_1q^h, the congruence class $l \pmod{k}$ is the union of classes $l^{(j)} \pmod{k_1q^h}$, $j = 1, \ldots, q^h/k_2$. For each j write

$$l^{(j)} = l_1^{(j)} + q^h l_2^{(j)}$$

where $0 \le l_1^{(j)} < q^h$. Then

$$A(x|k,l,t) = \sum_{j=1}^{q^h/k_2} A\left(\frac{x - l_1^{(j)}}{q^h}\middle| k_1, l_2^{(j)}, t - f(l_1^{(j)}).\right)$$

Since $(k_1, q) = 1$, we are led to the next reduction.

Reduction 2. Suppose that $(k, q) = 1$. Let $k = k_1k_2$ where k_1 is the largest divisor of k so that $(k_1, q-1) = 1$. Then there is a positive constant $c_3 = c_3(f)$ so that

$$A(x|k,l,t) = \frac{1}{k_1} A(x|k_2, l, t) + O\left(x^{1-\frac{c_3}{\log 2k}}\right).$$

Before moving to the next reduction, we note that the primes dividing k_2 must also divide $q - 1$.

Reduction 3. Suppose that the prime divisors of k also divide $q-1$. Then there is a positive constant $c_4 = c_4(f)$ so that

$$A(x|k,l,t) = \frac{(k,q-1)}{k} A(x|(k,q-1),l,t) + O\left(x^{1-\frac{c_4}{\log 2k}}\right).$$

We have therefore reduced to the case in which the modulus is a divisor of $q-1$.

Reduction 4. We now come to the first new reduction. The proof closely follows that of Reductions 2 and 3 of [6], substituting our Lemma 1 for their Lemma 1 and our Lemma 2 for their Lemma 3. We therefore choose to omit it. Suppose that $k|q-1$ and let d denote the greatest common divisor of $j_1 f(j_2) - j_2 f(j_1)$ for $j_1, j_2 \in \{0, 1, \ldots, q-1\}$. Then there is a positive constant $c_5 = c_5(f)$ so that

$$A(x|k,l,t) = \frac{(k,d)}{k} A(x|(k,d),l,t) + O\left(x^{1-c_5}\right).$$

We note that in the case $f = s_q$ we have $d = 0$, making this reduction unnecessary. As such, it has no analogue in [6].

Reduction 5. Suppose that k divides $(q-1,d)$. Then $q^j \equiv 1 \pmod{k}$ for all $j \geq 1$ and $j_1 f(j_2) \equiv j_2 f(j_1) \pmod{k}$ for all $0 \leq j_1, j_2 \leq q-1$. From this and the complete q-additivity of f it follows that $mf(n) \equiv nf(m) \pmod{k}$ for all $m, n \in \mathbb{Z}_0^+$. Let $F = (f(1), f(2), \ldots, f(q-1))$ and suppose further that $(F, q-1) = 1$. Since the values of f are linear combinations of $f(1), f(2), \ldots, f(q-1)$ with nonnegative integer coefficients, the condition

$(F, q-1) = 1$ implies that we can find an $m \in \mathbb{Z}^+$ so that $(f(m), q-1) = 1$. Therefore if $n \in \mathbb{Z}_0^+$ satisfies $f(n) = t$ we have $mt = mf(n) \equiv nf(m) \pmod{k}$ and it follows that

$$A(x|k,l,t) = \begin{cases} a(x|t) & \text{if } mt \equiv lf(m) \pmod{k}, \\ 0 & \text{otherwise.} \end{cases}$$

This is the analogue of the final reduction in section 4.5 of [6].

4.2 Unrestricted argument

We now turn to the distribution of the values of f when its argument is free to take on any value. We let m and σ denote the mean and standard deviation of f on the set $D = \{0, 1, \ldots, q-1\}$ of base q digits. That is,

$$m = \frac{1}{q}\sum_{j=0}^{q-1} f(j) \ , \ \sigma^2 = \frac{1}{q}\sum_{j=0}^{q-1} f(j)^2 - m^2.$$

We also set

$$N_x = \left\lfloor \frac{\log x}{\log q} \right\rfloor$$

where $\lfloor y \rfloor$ is the greatest integer not exceeding y.

Following [1], we view D as a probability space in which each element is assigned a measure of $1/q$. We use the digits of the base q expansion (1) to identify the set of nonnegative integers n strictly less than q^N ($N \in \mathbb{Z}^+$) with the product space D^N. In this way f can be viewed as the random variable on D^N obtained by summing together N independent copies of f

acting on D alone. Applying Theorem 15, Chapter III of [9] in this setting, it is not difficult to deduce the next result.

Proposition 1. *Let* $\epsilon \in (0, 1/2)$ *and set* $I = [mN_x - N_x^{1/2+\epsilon}, mN_x + N_x^{1/2+\epsilon}]$. *Then*

$$\#\{0 \leq n < x | f(n) \notin I\} \ll \frac{x}{(\log x)^2}$$

the implied constant depending only on f *and* ϵ.

We remark that in the case $f = s_q$, this proposition follows from Lemma 4 of [8]. In fact, using the line of reasoning we have suggested, it is possible to obtain a more general result very similar to that lemma in which $N_x^{1/2+\epsilon}$ is replaced by $N_x^{1/2}\lambda_x$, where $\lambda_x = o(N_x^{1/2})$.

As observed in [6] in the case $f = s_q$, Theorem 6, Chapter VII of [9] can be used to obtain the following result.

Proposition 2. *If* $(f(1), f(2), \ldots, f(q-1)) = 1$ *then uniformly for* $t \in \mathbb{Z}$ *we have*

$$a(x|t) = \frac{x}{\sigma\sqrt{N_x}}\varphi\left(\frac{t - mN_x}{\sigma\sqrt{N_x}}\right) + O\left(\frac{x}{N_x}\right)$$

where $\varphi(y) = (2\pi)^{-1/2}e^{-y^2/2}$. *The implied constant depends only on* f.

Corollary 1. *If* $F = (f(1), f(2), \ldots, f(q-1))$ *then*

$$a(x|t) = \frac{Fx}{\sigma\sqrt{N_x}}\varphi\left(\frac{t - mN_x}{\sigma\sqrt{N_x}}\right) + O\left(\frac{x}{N_x}\right)$$

when $t \equiv 0 \pmod{F}$ *and* $a(x|t) = 0$ *otherwise.*

Proof. Apply the proposition to the function $g = f/F$. □

5 Proof of Theorem 1

It suffices to consider the case in which $m \geq 0$, since $N_f(x) = N_{-f}(x)$. Fixing $\epsilon \in (0, 1/2)$, according to Proposition 1 we have

$$N_f(x) = \sum_{\substack{t \in I \\ t \equiv 0 \pmod{F}}} A(x||t|, 0, t) + O\left(\frac{x}{(\log x)^2}\right). \tag{6}$$

As in Section 5 of [6], one can show that Reductions 1 through 5 together with the corollary to Proposition 2 yield

$$A(x||t|, 0, t) = \frac{(t, q-1, d)}{|t|} a(x|t) + O\left(\frac{x \log N_x}{t N_x}\right) \tag{7}$$

uniformly for (nonzero) $t \in I$. Since $(t, q-1, d)$ depends only on the residue of $t \pmod{q-1}$, substitution of (7) into (6) yields

$$N_f(x) = \sum_{j=1}^{q-1} (j, q-1, d) \sum_{\substack{t \in I \\ t \neq 0 \\ t \equiv 0 \pmod{F} \\ t \equiv j \pmod{q-1}}} \frac{a(x|t)}{|t|} + O\left(\frac{x \log N_x}{N_x} E_1(x)\right) \tag{8}$$

where

$$E_1(x) = \begin{cases} N_x^{\epsilon - 1/2} & \text{if } m > 0, \\ \log N_x & \text{if } m = 0. \end{cases}$$

Lemma 3. *For $j = 1, 2, \ldots, q-1$ we have*

$$\sum_{\substack{t \in I \\ t \neq 0 \\ t \equiv 0 \pmod{F} \\ t \equiv j \pmod{q-1}}} \frac{a(x|t)}{|t|} = \frac{1}{q-1} \sum_{\substack{t \in I \\ t \neq 0 \\ t \equiv 0 \pmod{F}}} \frac{a(x|t)}{|t|} + O\left(\frac{x}{N_x} E_2(x)\right)$$

where

$$E_2(x) = \begin{cases} N_x^{\epsilon - 1/2} & \textit{if } m > 0, \\ N_x^{1/2} & \textit{if } m = 0. \end{cases}$$

Proof. Since we have assumed $(F, q-1) = 1$, for each $j = 1, 2, \ldots, q-1$ there is a unique $b_j \pmod{F(q-1)}$ so that the two simultaneous congruences $t \equiv 0 \pmod{F}$ and $t \equiv j \pmod{q-1}$ are equivalent to the single congruence $t \equiv b_j \pmod{F(q-1)}$. The corollary to Proposition 2 implies that for $t, k \equiv 0 \pmod{F}$ we have

$$\frac{a(x|t+k)}{|t+k|} = \frac{a(x|t)}{|t|} + O\left(\frac{x}{|t|\log x} + \frac{x}{t^2(\log x)^{1/2}}\right) \tag{9}$$

when $t + k \neq 0$, the implied constant depending only on f and k.

When $m > 0$ we therefore have

$$\begin{aligned}
\sum_{\substack{t\in I\\ t\neq 0\\ t\equiv 0 \pmod{F}\\ t\equiv j \pmod{q-1}}} \frac{a(x|t)}{|t|} &= \sum_{\substack{t\in I\\ t\equiv b_j \pmod{F(q-1)}}} \frac{a(x|t)}{t} \\
&= \frac{1}{q-1} \sum_{\substack{t\in I\\ t\equiv b_j \pmod{F(q-1)}}} \sum_{k=0}^{q-2} \frac{a(x|t+kF)}{t+kF} \\
&\quad + O\left(\frac{x}{N_x^{3/2-\epsilon}}\right) \\
&= \frac{1}{q-1} \sum_{\substack{s\in I\\ s\equiv 0 \pmod{F}}} \frac{a(x|s)}{s} + O\left(\sum_{\substack{s\in J\\ s\equiv 0 \pmod{F}}} \frac{a(x|s)}{s} \right) \\
&\quad + O\left(\frac{x}{N_x^{3/2-\epsilon}}\right)
\end{aligned}$$

where J is the set of integers with distance at most qF from the endpoints of I. Since $a(x|s) \ll x/N_x^{1/2}$, the sum over J contributes no more than the second error term, proving the lemma in this case.

When $m = 0$ we may carry through the same analysis, being careful in the second step to omit from the summation the single value of t for which

$t + kF = 0$. This introduces an error of size $O(x/N_x^{1/2})$, which is consistent with the statement of the lemma. □

Combining Lemma 3 with equation (8) we can now complete the proof. When $m > 0$ we have

$$\begin{aligned}
N_f(x) &= \left(\frac{1}{q-1}\sum_{j=1}^{q-1}(j, q-1, d)\right) \sum_{\substack{t \in I \\ t \equiv 0 \pmod{F}}} \frac{a(x|t)}{t} + O\left(\frac{x \log N_x}{N_x^{3/2-\epsilon}}\right) \\
&= \frac{1}{mN_x}\left(\frac{1}{q-1}\sum_{j=1}^{q-1}(j, q-1, d)\right) \sum_{\substack{t \in I \\ t \equiv 0 \pmod{F}}} a(x|t) \\
&\quad + O\left(\frac{x \log\log x}{(\log x)^{3/2-\epsilon}}\right) \\
&= \frac{\log q}{m \log x}\left(\frac{1}{q-1}\sum_{j=1}^{q-1}(j, q-1, d)\right) \sum_{\substack{t \in I \\ t \equiv 0 \pmod{F}}} a(x|t) \\
&\quad + O\left(\frac{x \log\log x}{(\log x)^{3/2-\epsilon}}\right) \\
&= \frac{\log q}{m}\left(\frac{1}{q-1}\sum_{j=1}^{q-1}(j, q-1, d)\right) \frac{x}{\log x} + O\left(\frac{x \log\log x}{(\log x)^{3/2-\epsilon}}\right)
\end{aligned}$$

where in the final line we have used Proposition 1 to replace the sum with $x + O(x/(\log x)^2)$. Since $\epsilon \in (0, 1/2)$ was arbitrary, we may discard the $\log\log x$ term in the error, giving the stated result.

When $m = 0$ we have

$$N_f(x) = \left(\frac{1}{q-1}\sum_{j=1}^{q-1}(j, q-1, d)\right) \sum_{\substack{t \in I \\ t \neq 0 \\ t \equiv 0 \pmod{F}}} \frac{a(x|t)}{|t|} + O\left(\frac{x}{N_x^{1/2}}\right) \qquad (10)$$

Set $\epsilon = 1/4$ for convenience. Writing $t = Fs$ and using the corollary to

Proposition 2 we obtain

$$\sum_{\substack{t\in I \\ t\neq 0 \\ t\equiv 0 \pmod F}} \frac{a(x|t)}{|t|} = \frac{2x}{\sigma\sqrt{N_x}} \sum_{1\le s\le N_x^{3/4}/F} \frac{1}{s}\varphi\left(\frac{sF}{\sigma\sqrt{N_x}}\right) + O\left(\frac{x\log N_x}{N_x}\right). \quad (11)$$

Now

$$\begin{aligned}
\sum_{1\le s\le N_x^{3/4}/F} \frac{1}{s}\varphi\left(\frac{sF}{\sigma\sqrt{N_x}}\right) &= \int_1^{N_x^{3/4}/F} \frac{1}{s}\varphi\left(\frac{sF}{\sigma\sqrt{N_x}}\right) ds + O(1) \\
&= \frac{1}{\sqrt{2\pi}} \int_{F/\sigma\sqrt{2N_x}}^{N_x^{1/4}/\sigma\sqrt{2}} \frac{e^{-u^2}}{u} du + O(1) \\
&= \frac{1}{\sqrt{2\pi}} \int_{F/\sigma\sqrt{2N_x}}^{1} \frac{e^{-u^2}}{u} du + O(1) \\
&= \frac{1}{\sqrt{2\pi}} \int_{F/\sigma\sqrt{2N_x}}^{1} \frac{1}{u} du + O(1) \\
&= \frac{1}{2\sqrt{2\pi}} \log N_x + O(1) \\
&= \frac{1}{2\sqrt{2\pi}} \log\log x + O(1).
\end{aligned}$$

Returning to equations (10) and (11) we find that

$$\begin{aligned}
N_f(x) &= \frac{1}{(2\pi\sigma^2)^{1/2}} \left(\frac{1}{q-1}\sum_{j=1}^{q-1}(j,q-1,d)\right) \frac{x\log\log x}{N_x^{1/2}} + O\left(\frac{x}{N_x^{1/2}}\right) \\
&= \left(\frac{\log q}{2\pi\sigma^2}\right)^{1/2} \left(\frac{1}{q-1}\sum_{j=1}^{q-1}(j,q-1,d)\right) \frac{x\log\log x}{(\log x)^{1/2}} \\
&\quad + O\left(\frac{x}{(\log x)^{1/2}}\right)
\end{aligned}$$

which concludes the proof.

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