On the Counting Function of the Generalized Niven Numbers

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Niven Numbers

A Niven number is a number whose sum of digits divides itself.

A base q Niven number is a number n whose sum of digits base q divides the number n.

Counting Function of Niven Numbers

- Introduced by Ivan Niven at a lecture in 1977.
- $N_q(x)$ is the number of base q Niven numbers less than x
- Kennedy and Cooper show the natural density of Niven numbers is 0

Asymptotic Formula Counting Function

De Koninck, Doyon and Kátai in 2003 showed

$$N_q(x) = (c_q + o(1))\frac{x}{\log x}$$

where

$$c_q = \frac{2\log q}{(q-1)^2}\sum_{j=1}^{q-1}(j, q-1)$$

Asymptotic Formula Counting Function

Independently proven by Mauduit, Pomerance and Sarkozy (with an improvement in the error term)

$$N_q(x) = c_q \frac{x}{\log x} + O\left(\frac{x}{(\log x)^{\frac{9}{8}}}\right)$$

In our work we improve the exponent in the log of the error term from $\frac{9}{8}$ to $(\frac{3}{2} - \epsilon)$

Completely q-Additive Functions

Let $q \geq 2$ be a fixed integer and let f be an arbitrary complex-valued function defined on the set of nonnegative integers.

We say that f is completely q-additive if

$$f(aq^j + b) = f(a) + f(b)$$

for all nonnegative integers a, b, j satisfying $b < q^j$

Completely q-Additive Functions

A function f is completely q-additive if and only if $f(0) = 0$ and

$$f(n) = f\left(\sum_{j=0}^{\infty} a_j(n) q^j\right) = \sum_{j=0}^{\infty} f(a_j(n))$$

Sum of digits base q is an example of a completely q-additive function

Generalized Niven Numbers

A number n is an f-Niven number if $f(n)$ divides n.

Let $N_f(x)$ be the number of f-Niven numbers $n \leq x$.

What can we say about this counting function?

Asymptotic Formula for Generalized Niven Numbers

Let f be an arbitrary non-zero, integer-valued, completely q-additive function and set

$$m = \frac{1}{q}\sum_{j=0}^{q-1} f(j),$$

$$\sigma^2 = \frac{1}{q}\sum_{j=1}^{q-1} f(j)^2 - m^2,\ F = \big(f(1), f(2), \ldots, f(q-1)\big)$$ and

assume $(F, q-1) = 1$.

Asymptotic Formula for Generalized Niven Numbers

Let $d = gcd\{r\, f(s) - s\, f(r) \mid r, s \in \{0, 1, \ldots, q-1\}\}$

If $m \neq 0$ then for any $\epsilon \in (0, \frac{1}{2})$

$$N_f(x) = c_f \frac{x}{\log x} + O\left(\frac{x}{(\log x)^{\frac{3}{2}-\epsilon}}\right)$$

where

$$c_f = \frac{\log q}{|m|}\left(\frac{1}{q-1}\sum_{j=1}^{q-1}(j, q-1, d)\right)$$

Asymptotic Formula for Generalized Niven Numbers

If $m = 0$ then

$$N_f(x) = c_f \frac{x \log\log x}{(\log x)^{\frac{1}{2}}} + O\left(\frac{x}{(\log x)^{\frac{1}{2}}}\right)$$

where

$$c_f = \left(\frac{\log q}{2\pi\sigma^2}\right)^{\frac{1}{2}} \left(\frac{1}{q-1} \sum_{j=0}^{q-1} (j, q-1, d)\right)$$

Outline of Proof

Let $A(x|k,l,t) = \#\{0 \le n \le x \mid n \equiv l \mod k, f(n) = t\}$
$\alpha(x|t) = \#\{0 \le n \le x \mid f(n) = t\}$

$$N_f(x) = \sum_{t=-\infty}^{\infty} A(x||t|,0,t)$$

- Change interval in the sum to make it more manageable
- Reduce $A(x||t|,0,t)$ to $\alpha(x|t)$
- Use the Central Limit Theorem on $\alpha(x|t)$

Outline of Proof

Reducing $A(x\,\|\,k, l, t)$ to $\alpha(x\,|\,t)$ takes five steps:

- Let $k = t_1 k_3$ with t_1 the largest integer such that $(t_1, q) = 1$. Now you can transfrom the problem of $A(x\,|\,k, l, t)$ to $A(x\,|\,t_1, l, t)$
- Let $t_1 = k_1 k_2$ with k_1 largest such that $(k_1, q-1) = 1$. Now you can transform $A(x\,|\,t_1, l, t)$ to $A(x\,|\,k_2, l, t)$
- Transform from $A(x\,|\,k_2, l, t)$ to $A(x\,|\,(k, q-1), l, t)$
- Transform from $A(x\,|\,(k, q-1), l, t)$ to $A(x\,|\,(k, q-1, d), l, t)$
- Transform from $A(x\,|\,(k, q-1, d), l, t)$ to $\alpha(x\,|\,t)$ using that $mf(n) \equiv nf(m) \mod a$ if $a\,|\,(q-1, d)$

Outline of Proof

Reducing $A(x \| k, l, t)$ to $\alpha(x | t)$ takes five steps:

- Let $k = t_1 k_3$ with t_1 the largest integer such that $(t_1, q) = 1$. Now you can transfrom the problem of $A(x | k, l, t)$ to $A(x | t_1, l, t)$
- Let $t_1 = k_1 k_2$ with k_1 largest such that $(k_1, q-1) = 1$. Now you can transform $A(x | t_1, l, t)$ to $A(x | k_2, l, t)$
- Transform from $A(x | k_2, l, t)$ to $A(x | (k, q-1), l, t)$
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Outline of Proof

Reducing $A(x||k, l, t)$ to $\alpha(x|t)$ takes five steps:

- Let $k = t_1 k_3$ with t_1 the largest integer such that $(t_1, q) = 1$. Now you can transfrom the problem of $A(x|k, l, t)$ to $A(x|t_1, l, t)$
- Let $t_1 = k_1 k_2$ with k_1 largest such that $(k_1, q - 1) = 1$. Now you can transform $A(x|t_1, l, t)$ to $A(x|k_2, l, t)$
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Outline of Proof

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Examples

- If f is the sum of digits, then we have the asymptotic proven by De Koninck, Doyon and Kátai
- If $f(x) = \begin{cases} 1 & \text{if } x = 1 \\ 0 & \text{if } x \neq 1 \text{ and } x < q \end{cases}$
  The asymptotics for this would be $q \log q \dfrac{x}{\log x}$

Examples

If f is the sum of the squares of the digits, then the asymptotic formula for its counting function is

- $\dfrac{6\log q}{(q-1)(2q-1)}\dfrac{x}{\log x}$ if q is even
- $\dfrac{9\log q}{(q-1)(2q-1)}\dfrac{x}{\log x}$ if q is odd