On Egyptian fractions of length 3

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joint work with
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Egyptian Fractions

Any positive rational a/n can be written as the sum of positive unit fractions

$$\frac{a}{n} = \frac{1}{m_1} + \frac{1}{m_2} + \cdots + \frac{1}{m_k}.$$

The above is an example of an Egyptian fraction decomposition of length k.

Erdős–Straus conjecture

Conjecture

There exist positive integers m_1, m_2, m_3 such that

$$\frac{4}{n} = \frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3}.$$

- Suffices to check it for $n = p$.
- True for $p \equiv 3 \mod 4$ because

$$\frac{4}{p} = \frac{1}{\frac{p+1}{4}} + \frac{1}{\frac{p+1}{2}p} + \frac{1}{\frac{p+1}{2}p}.$$

- It has been verified for $n \leq 10^{14}$.
- The set of exceptions has density 0. (Vaughan 1970)

Averaging

Since we can't prove it, let's average!

Theorem (Elshotz–Tao (2013))

Let

$$f(n) = \#\left\{(m_1, m_2, m_3) \in \mathbb{N}^3 \,:\, \frac{4}{n} = \frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3}\right\},$$

then

$$x \log^2 x \ll \sum_{p \le x} f(p) \ll x \log^2 x \, \log\log x.$$

Functions we want to estimate

$$A_k(n) = \#\left\{a \in \mathbb{N} \,:\, \frac{a}{n} = \frac{1}{m_1} + \frac{1}{m_2} + \cdots + \frac{1}{m_k}\right\}.$$

$$A_k^*(n) = \#\left\{a \in \mathbb{N} \,:\, \gcd(a,n) = 1, \frac{a}{n} = \frac{1}{m_1} + \frac{1}{m_2} + \cdots + \frac{1}{m_k}\right\}.$$

$$f_a(n) = \#\left\{(m_1, m_2, m_3) \in \mathbb{N}^3 \,:\, \frac{a}{n} = \frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3}\right\}.$$

$$F(n) = \#\left\{(a, m_1, m_2, m_3) \in \mathbb{N}^4 \,:\, \frac{a}{n} = \frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3}\right\}.$$

Length 2

$$A_k(n) = \#\left\{a \in \mathbb{N} : \frac{a}{n} = \frac{1}{m_1} + \frac{1}{m_2} + \cdots + \frac{1}{m_k}\right\}.$$

$$A_k^*(n) = \#\left\{a \in \mathbb{N} : \gcd(a,n) = 1, \frac{a}{n} = \frac{1}{m_1} + \frac{1}{m_2} + \cdots + \frac{1}{m_k}\right\}.$$

Theorem (Croot, Dobbs, Friedlander, Hetzel, Pappalardi (2000))

For any $\varepsilon > 0$,

$$A_2^*(n) \ll n^{\varepsilon}, \qquad A_2(n) \ll n^{\varepsilon}.$$

Furthermore

$$x \log^3 x \ll \sum_{n \le x} A_2^*(n) \ll x \log^3 x,$$

and

$$x \log^4 x \ll \sum_{n \le x} A_2(n) \ll x \log^4 x.$$

Length 3, averaging over primes

$$A_k(n) = \#\left\{a \in \mathbb{N} : \frac{a}{n} = \frac{1}{m_1} + \frac{1}{m_2} + \cdots + \frac{1}{m_k}\right\}.$$

Theorem (Luca–Pappalardi (2019))

$$x \log^3 x \ll \sum_{p \le x} A_3(p) \ll x \log^5 x.$$

$A_3(n)$

Theorem (Croot, Dobbs, Friedlander, Hetzel, Pappalardi (2000))

For any $\varepsilon > 0$*,*

$$A_3(n) \ll n^{\frac{1}{2}+\varepsilon}.$$

Theorem (Banderier, Gómez Ruiz, Luca, Pappalardi, Treviño)

Let $h(n) = C/\log\log n$*, where* $C = \frac{2\log(48)\log(\log(6983776800))}{\log(6983776800)} \approx 1.066$*. Then*

$$A_3(n) \leq 10n^{\frac{1}{2}+\frac{13}{4}h(n)}\log n.$$

Corollary

For $n \geq 10^{10^{23}}$*,*

$$A_3(n) \leq \frac{1}{100}n^{\frac{1}{2}+\frac{1}{15}}.$$

$f_a(n)$

$$f_a(n) = \#\left\{(m_1, m_2, m_3) \in \mathbb{N}^3 \,:\, \frac{a}{n} = \frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3}\right\}.$$

Theorem (Elshotz–Tao (2013))

$$f_4(p) \ll p^{\frac{3}{5}+o(1)}.$$

Theorem (Banderier, Gómez Ruiz, Luca, Pappalardi, Treviño)

For any ρ,

$$f_a(n) \leq n^{\varepsilon}\left(\frac{n^{\frac{1}{2}+\frac{\rho}{2}}}{a} + n^{1-\rho}\right).$$

Therefore,

$$f_a(n) \ll \frac{n^{\frac{2}{3}+\varepsilon}}{a^{\frac{2}{3}}}.$$

Explicit $f_a(n)$

Theorem (Banderier, Gómez Ruiz, Luca, Pappalardi, Treviño)

For any ρ, Let $1/3 \leq \rho$, and $n \geq 11000$. Then

$$f_a(n) \leq 6n^{5h(n)} \left(6\sqrt{2}\, \frac{n^{1/2+\rho/2}}{a} 10^{h(n)} + \frac{3}{2} n^{1-\rho} \log n \, 6^{h(n)} \right). \qquad (1)$$

Corollary

If $n \geq 10^{10^{23}}$, then

$$f_a(n) < \frac{1}{100} n^{\frac{1}{10}} \left(\frac{n^{1/2+\rho/2}}{a} + n^{1-\rho} \right).$$

$F(n)$

$$F(n) = \#\left\{(a, m_1, m_2, m_3) \in \mathbb{N}^4 \;:\; \frac{a}{n} = \frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3}\right\}.$$

Theorem (Banderier, Gómez Ruiz, Luca, Pappalardi, Treviño)

Let $\varepsilon > 0$, then

$$F(n) \ll n^{\frac{5}{6}+\varepsilon}.$$

This implies that for large enough n, $F(n) < n$. This suggests the question, what is the largest n such that $F(n) \geq n$.
The first values for which $F(n) < n$ are:
$F(8821) = 8590, F(11161) = 10270, F(11941) = 10120.$

Explicit $F(n)$

Theorem (Banderier, Gómez Ruiz, Luca, Pappalardi, Treviño)

For $n \geq 10^{10^{23}}$*.*

$$F(n) \leq \frac{1}{10} n$$

Parametrization Lemma

Lemma (Luca, Pappalardi)

Consider an Egyptian fraction decomposition of the irreducible fraction a/n:

$$\frac{a}{n} = \frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} \qquad \textit{with } \gcd(a, n) = 1 \tag{2}$$

Then there exist integers $D_1, D_2, D_3, v_1, v_2, v_3$ with

(i) $lcm(D_1, D_2, D_3) \mid n$ *and* $\gcd(D_1, D_2, D_3) = 1$*;*

(ii) $av_1 v_2 v_3 \mid D_1 v_1 + D_2 v_2 + D_3 v_3$ *and* $\gcd(v_i, D_j v_j) = 1$ *when* $i \neq j$*,*

and the denominators of the Egyptian fractions are given by

$$m_i = \frac{n(D_1 v_1 + D_2 v_2 + D_3 v_3)}{aD_i v_i}. \tag{3}$$

Conversely, if conditions (i)–(ii) are fulfilled, then the m_i's defined via (3) *are integers, and denominators of k unit fractions summing to a/n.*

Thank you!