ON SETS WHOSE SUBSETS HAVE INTEGER MEAN

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Abstract

We call a finite set of positive integers balanced if all its subsets have integer mean. For a positive integer N, let $M(N)$ be the cardinality of the largest balanced set all of whose elements are less than or equal to N, and let $S(N)$ be the cardinality of the balanced set with elements less than or equal to N that has maximal sum. We study properties of balanced sets and answer questions regarding what positive integers N satisfy $M(N) = S(N)$.

1. Introduction

We say a finite set A of positive integers is balanced if, for any subset $B \subseteq A$, the arithmetic mean of the elements of B is an integer, i.e.,

$$\frac{1}{|B|}\sum_{i\in B} i \in \mathbb{Z}.$$

The definition was introduced in problem 2 of the 31st Mexican Mathematical Olympiad held in 2017. The original problem asked to find the largest sum a balanced set whose maximal element is 2017 can have. The answer, 12859, is achieved with a balanced set with maximal number of elements less than or equal to 2017; i.e., it occurs with the set $\{2017, 1957, 1897, 1837, 1777, 1717, 1657\}$, which has 7 elements, and there are no balanced sets of 8 elements with maximal element at most 2017. I was a grader for this problem and some students would find the largest balanced set and assumed it had maximal sum, without proving it. To explain why these students should not get full credit, I gave the example of considering changing the number 2017 for 3000. With 3000, there is a balanced set with 8 elements, namely $\{3000, 2580, 2160, 1740, 1320, 980, 480, 60\}$, but it has a smaller sum than the balanced set with 7 elements $\{3000, 2940, 2880, 2820, 2760, 2700, 2640\}$. Therefore, having a larger number of elements does not guarantee having a larger sum (even when optimizing for largest possible elements). This led me to think of the question of which N satisfy that the largest balanced set with elements less than or equal to N is the one with the largest sum.

For a positive integer N, let $M(N)$ be the size of the largest balanced set all of whose elements are at most N. Let $S(N)$ be the size of the set with maximal sum among balanced sets all of whose elements are at most N. Our question is, for which N is $M(N) = S(N)$? For example $M(2017) = S(2017)$, yet $M(3000) \neq S(3000)$. Using a computer, we can verify that if $N \leq 1000000$, then $M(N) = S(N)$ for

$$\begin{aligned} 1 &\leq N \leq 18, \\ 31 &\leq N \leq 48, \\ 85 &\leq N \leq 300, \\ 571 &\leq N \leq 2940, \\ 18481 &\leq N \leq 22680, \\ 54181 &\leq N \leq 304920. \end{aligned}$$

If we let $L(n)$ be the least common multiple of $\{1, 2, \ldots, n\}$, then we can see that $18 = 3L(3)$, $48 = 4L(4)$, $300 = 5L(5)$, $2940 = 7L(7)$, $22680 = 9L(9)$, and $304920 = 11L(11)$. In other words, the upper bounds on these intervals are all of the form $mL(m)$. If we verify using a computer for which $m \leq 800$ is $M(mL(m)) = S(mL(m))$, we find that it works for all primes less than or equal to 800, along with 4, 9, and 121. This suggests that $M(pL(p)) = S(pL(p))$ for any prime p. Indeed, we can prove that.

Theorem 1. *Let p be prime. Then $M(pL(p)) = S(pL(p))$. Furthermore, $M(pL(p)+1) \neq S(pL(p)+1)$.*

The computational evidence also suggests that if $m > 121$ with $M(mL(m)) = S(mL(m))$, then m is prime. The following theorem partially proves it, by showing that if $M(mL(m)) = N(mL(m)$, then m is a prime power.

Theorem 2. *If m is not a prime power, then $M(mL(m)) \neq S(mL(m))$.*

Attacking prime powers is harder. However, with some nice work [1, 4] on large gaps between consecutive primes, we can prove that for large enough m, if $S(mL(m)) = M(mL(m))$, then m cannot be a prime power with exponent at least 3.

Theorem 3. *For $m \geq 10^{10^{15}}$ of the form q^k for a prime q and an exponent $k \geq 3$, then $M(mL(m)) \neq S(mL(m))$.*

Using results on large prime gaps [2] assuming the Generalized Riemann Hypothesis (GRH), we can prove

Theorem 4. *Assuming GRH, if $m = q^k$ for a prime q and exponent $k \geq 3$, then $M(mL(m)) \neq S(mL(m))$.*

That large squares of primes fail is harder to prove. The reason is that our proofs above depend on the existence of a prime $p \in [x - c_k x^{1-1/k}, x]$ for a constant c_k depending on k. However, for squares, we would need a prime in the interval $[x - c_2\sqrt{x}, x]$. The best unconditional result is due to Baker, Harman, and Pintz [1] who proved that for large enough x there is a prime in the interval $[x - x^{21/40}, x]$, while the best result under GRH, proved by Carneiro, Milinovich, and Soundararajan [2], is that for $x \geq 4$ there is a prime in the interval $[x - \frac{22}{25}\sqrt{x}\log x, x]$. To remove squares of primes we would need stronger conjectures regarding prime gaps. In particular, Crámer [3] conjectured that for large enough x there is a prime in the interval $[x - C\log^2 x, x]$, for some constant C. This assumption would be enough to prove that for large p, $M(p^2L(p^2)) \neq S(p^2L(p^2))$.

The above results describe what happens at the right endpoints of the intervals. The left endpoints are harder to estimate, but we can do enough to prove results about the natural density of the set of N for which $M(N) = S(N)$. In fact, we can prove the density does not exist because of the following theorem:

Theorem 5. *The set of N for which $M(N) = S(N)$ has upper density equal to 1, and it has lower density equal to 0.*

2. Balanced sets with maximal element N

An important property that balanced sets A with n elements satisfy is that the elements of A must all be congruent modulo $L(n-1)$. The following lemma proves this and evaluates the maximal sum a balanced set with m elements can have if all the elements are at most N.

Lemma 1. *For an integer $m \geq 3$, a balanced set A with m elements must satisfy that all of its elements are congruent modulo $L(m-1)$. Furthermore, the balanced set with m elements less than or equal to N whose sum is maximal is $\{N, N-L(m-1), N-2L(m-1), \ldots, N-(m-1)L(m-1)\}$. The maximal sum is*

$$mN - \frac{m(m-1)}{2}L(m-1).$$

Proof. Let $A = \{a_1, a_2, \ldots, a_m\}$ be a balanced set with m elements. Let $k \leq m-1$. Then $a_{i_1} + a_{i_2} + \ldots + a_{i_{k-1}} + a_{i_k} \equiv a_{i_1} + a_{i_2} + \ldots + a_{i_{k-1}} + a_{i_{k+1}} \bmod k$ for any indices $i_1, i_2, \ldots, i_k, i_{k+1}$. Therefore $a_{i_k} \equiv a_{i_{k+1}} \bmod k$. Since it is true for any indices, then $a_1 \equiv a_2 \equiv \ldots \equiv a_m \bmod k$. Because this is true for all $k \leq m-1$, then the elements are congruent modulo the least common multiple of $\{1, 2, \ldots, m-1\}$, which is $L(m-1)$. The set with elements less than or equal to N that has its elements as large as possible and congruent modulo $L(m-1)$ is

$$\{N, N-L(m-1), N-2L(m-1), \ldots, N-(m-1)L(m-1)\}.$$

We need to show this set is balanced. Since the elements are congruent modulo $L(m-1)$ than for any proper subset, the arithmetic mean is an integer. To conclude, we need to show the arithmetic mean of the set itself is an integer. The mean is

$$N - \frac{m(m-1)}{2m}L(m-1) = N - \frac{m-1}{2}L(m-1).$$

Since $L(m-1)$ is even for $m \geq 3$, the arithmetic mean is an integer. □

The next lemma allows us to determine the value of $M(N)$.

Lemma 2. *Let $m \geq 3$ be a positive integer. Then $M(N) = m$ if and only if $(m-1)L(m-1) + 1 \leq N \leq mL(m)$.*

Proof. If $N \leq (m-1)L(m-1)$, then we cannot have a balanced set with m elements since the m elements would need to be congruent modulo $L(m-1)$, but there are not m elements congruent to $L(m-1)$ less than or equal to $(m-1)L(m-1)$. For $N \geq (m-1)L(m-1)+1$, the set $\{N, N-L(m-1), \ldots, N-(m-1)L(m-1)\}$ is balanced. Therefore $M(N) \geq m$. For $N \leq mL(m)$, we cannot have a balanced set with $m+1$ elements. Therefore $M(N) \leq m$. Hence $M(N) = m$. □

We can also prove that M and S are monotone increasing functions.

Lemma 3. *The functions M and S are monotone increasing.*

Proof. That M is monotone increasing follows immediately from Lemma 2. For S, suppose $s = S(N+1) < S(N) = m$. Since $s < m$ and $S(N) = m$, we have from Lemma 1

$$mN - \frac{m(m-1)}{2}L(m-1) \geq sN - \frac{s(s-1)}{2}L(s-1).$$

But, since $m > s$, we have

$$m(N+1) - \frac{m(m-1)}{2}L(m-1) > s(N+1) - \frac{s(s-1)}{2}L(s-1).$$

Therefore $S(N+1) \neq s$. Contradiction! □

3. Proving theorems involving $N = mL(m)$

The first theorem to prove concerns showing $M(pL(p)) = N(pL(p))$.

Proof of Theorem 1. First we will prove $M(pL(p)) = S(pL(p))$. Consider balanced sets whose maximal element is at most $pL(p)$. By Lemma 2, we have $M(pL(p)) = p$.

By Lemma 1, the maximal sum of a balanced set with p elements less than or equal to $pL(p)$ is

$$p^2L(p) - \frac{p(p-1)}{2}L(p-1).$$

Since p is prime, $L(p-1) = L(p)/p$. Therefore, the sum is

$$p^2L(p) - \frac{p-1}{2}L(p) = pL(p)\left(p - \frac{1}{2} + \frac{1}{2p}\right) > pL(p)(p-1).$$

A balanced set with less than p elements less than or equal to $pL(p)$ has sum smaller than $pL(p)(p-1)$. Therefore, $M(pL(p)) = S(pL(p))$.

Now, let us prove $M(pL(p)+1) > S(pL(p)+1)$. By Lemma 2, $M(pL(p)+1) = p+1$. By Lemma 1, the maximal sum of a balanced set with $p+1$ elements is

$$(p+1)(pL(p)+1) - \frac{p(p+1)}{2}L(p). \tag{1}$$

The maximal sum of a balanced set with p elements is

$$p(pL(p)+1) - \frac{p(p-1)}{2}L(p-1). \tag{2}$$

Using that $L(p-1) = L(p)/p$ for p prime, subtracting (1) from (2) yields

$$\frac{p(p+1)}{2}L(p) - \frac{p(p-1)}{2p}L(p) - pL(p) - 1.$$

But, since $p \geq 3$, we have

$$\frac{pL(p)}{2}\left(p - 2 + \frac{1}{p}\right) - 1 > \frac{pL(p)}{2} - 1 > 0.$$

Therefore, $S(pL(p)+1) \leq p < p+1 = M(pL(p)+1)$. □

The next theorem to prove concerns showing $M(mL(m)) \neq S(mL(m))$ when m is not a prime power.

Proof of Theorem 2. By Lemma 2, we have $M(mL(m)) = m$. We want to show $S(mL(m)) < m$. Let p be the largest prime less than or equal to m. Then

$$L(p-1) = \frac{L(p)}{p} \leq \frac{L(m)}{p}.$$

By Bertrand's postulate [7, Theorem 8, p. 137], there is a prime between $\frac{m}{2}+1$ and m. Therefore, p satisfies $\frac{m}{2}+1 \leq p \leq m-1$.

Since m is not a power of a prime, $L(m-1) = L(m)$. Now, by Lemma 1, we have that the balanced set with maximal sum with m elements less than or equal to $mL(m)$ has sum

$$m^2L(m) - \frac{m(m-1)}{2}L(m-1) = m^2L(m) - \frac{m(m-1)}{2}L(m) = L(m)\left(\frac{m^2+m}{2}\right), \tag{3}$$

while the balanced set of maximal sum with p elements less than or equal to $mL(m)$ has sum

$$pmL(m) - \frac{p(p-1)}{2}L(p-1) \geq pmL(m) - \frac{p(p-1)}{2p}L(m) = pmL(m) - \frac{p-1}{2}L(m). \tag{4}$$

Since $mL(m) < L(m)$ for $m \geq 2$, the right side of (4) is increasing with p; therefore it is minimal when p is minimal. In particular, the maximal sum of a balanced set with p elements less than or equal to $mL(m)$ is greater than

$$\left(\frac{m}{2}+1\right)mL(m) - \frac{\frac{m}{2}+1-1}{2}L(m) = L(m)\left(\frac{m^2}{2}+m-\frac{m}{4}\right) > L(m)\left(\frac{m^2}{2}+\frac{m}{2}\right).$$

But this is greater than the sum with m elements; therefore $S(mL(m)) < m$. □

To prove results on prime powers, we will need better bounds on gaps of primes than Bertrand's postulate. Dudek [4] has the following nice result about the existence of primes between cubes.

Theorem 6. *For $m \geq e^{e^{33.3}}$, there exists a prime p such that $m^3 \leq p < m^3 + 3m^2$. In particular, there is a prime p such that*

$$m^3 < p < (m+1)^3.$$

For our proof we actually need the slightly tighter bound $m^3 - \frac{1}{3}m^2 < p < m^3$. The following lemma can be proved by following the steps performed by Dudek to prove Theorem 6. We will not include the proof, as it is a straightforward adaptation.

Lemma 4. *For all $m \geq 10^{10^{15}}$ there is a prime p such that*

$$m^3 - \frac{1}{3}m^2 < p < m^3.$$

We are now ready to prove that for large enough m, we cannot have $M(mL(m)) = S(mL(m)$ if m is a prime power with exponent greater than or equal to 3.

Proof of Theorem 3. We need only show $S(mL(m)) < m$. First, note that $L(m-1) = L(m)/q$. Now, let p be the largest prime less than or equal to m. By Lemma

4, $p > m - \frac{1}{3}m^{2/3}$. By Lemma 1, we have that the balanced set of m elements less than or equal to $mL(m)$ with maximal sum has sum

$$m^2L(m) - \frac{m(m-1)}{2}L(m-1) = m^2L(m) - \frac{m(m-1)}{2q}L(m), \tag{5}$$

while the one with p elements has sum

$$pmL(m) - \frac{p(p-1)}{2}L(p-1) \geq pmL(m) - \frac{p(p-1)}{2qp}L(m) = pmL(m) - \frac{p-1}{2q}L(m), \tag{6}$$

because $L(p-1) = \frac{L(p)}{p} \leq \frac{L(m-1)}{p} = \frac{L(m)}{pq}$.

Subtracting (5) from (6) yields that the difference of these two sums is at least

$$\frac{L(m)}{2q}\left(2qpm - p + 1 - 2qm^2 + m^2 - m\right).$$

Since $2qm > 0$ the expression is increasing with p, so using the bound $p > m - \frac{1}{3}m^{2/3}$ yields the bound

$$\frac{L(m)}{2q}\left(-\frac{2q}{3}m^{5/3} + \frac{1}{3}m^{2/3} + 1 + m^2 - 2m\right).$$

Since $k \geq 3$, then $q = m^{1/k} \leq m^{1/3}$. Therefore, the difference is at least

$$\frac{L(m)}{2q}\left(\frac{1}{3}m^2 + \frac{1}{3}m^{2/3} + 1 - 2m\right).$$

This last expression is positive for $m \geq 6$. □

It would be great to prove the above for all $m \geq 3$ and to classify the primes p such that $M(p^2L(p^2)) = S(p^2L(p^2))$, but this is hard. One thing we can do is prove the above for $m \geq 3$ assuming GRH, which is Theorem 4, and we can prove that for large enough p, $M(p^2L(p^2)) \neq S(p^2L(p^2))$ assuming Crámer's conjecture.

Proof of Theorem 4. From the proof of Theorem 3 we see that we need only show that there is a prime p in the interval $[m - \frac{1}{3}m^{2/3}, m]$. From Carneiro, Mininovich, and Soundararajan [2, Theorem 5] we know that assuming GRH, for $m \geq 4$, there is a prime in the interval $[m, m + \frac{22}{25}\sqrt{m}\log m]$. This implies that for $m \geq 8.1 \times 10^{11}$, there is a prime between $m - \frac{1}{3}m^{2/3}$ and m. We can verify with a computer program[1] that for $m \geq 223$, there is always a prime between $m - \frac{1}{3}m^{2/3}$ and m. Since we had checked all $m \leq 800$ with a computer and the only non-primes were 4, 9, and 121. The proof follows. □

[1] The computer program would do the following. Start with $a = 8.1 \times 10^{11}$, then compute $b = (1/3)a^{2/3}$. Find the first prime p above $a - b$. If it is smaller than or equal to a, then we have a prime in the interval. Otherwise you fail. Now let $a = p$ and repeat the process until it fails, which it does when $a = 211$.

4. Upper and lower density bounds

Let A be the set of all N such that $M(N) = S(N)$. That is,

$$A = \{N \in \mathbb{N} \,|\, M(N) = S(N)\}.$$

Let $A(x) = \{a \leq x \,|\, a \in A\}$ be the counting function of A. The upper density of A is

$$\delta^+ = \limsup_{x \to \infty} \frac{A(x)}{x},$$

and the lower density is

$$\delta^- = \liminf_{x \to \infty} \frac{A(x)}{x}.$$

To be able to prove our results about density, we will need the following lemma about an interval I where $M(N) = S(N)$ for all $N \in I$.

Lemma 5. *Suppose $q < p$ are consecutive primes for which there is no prime power in the interval (q, p). Let $k = p - q$.*

1. *For N such that*

$$pL(p) \geq N > \max\left\{\left(p - \frac{1}{2}\right) L(p-1), \frac{L(p)(p^2 - p - q + 1)}{2pk}\right\},$$

we have $M(N) = S(N)$.

2. *If*

$$\frac{L(p)(p^2 - p - q + 1)}{2pk} > \left(p - \frac{1}{2}\right) L(p-1),$$

for N satisfying

$$qL(q) + 1 \leq N < \frac{L(p)(p^2 - p - q + 1)}{2pk},$$

we have $M(N) \neq S(N)$.

Proof. For the first part, note that $\left(p - \frac{1}{2}\right) L(p-1) \geq (p-1)L(p-1) + 1$ since $p \geq 3$. Since $pL(p) \geq N \geq (p-1)L(p-1) + 1$, then $M(N) = p$. Therefore, for N, there exists a balanced set with p elements. Since there are no prime powers between q and p, we have

$$L(q) = L(q+1) = \ldots = L(p-1) = \frac{L(p)}{p}.$$

Suppose $q + 1 \leq m \leq p$. The maximal sum for a balanced set with m elements less than or equal to N is

$$mN - \frac{m(m-1)}{2} L(m-1) = mN - \frac{m(m-1)}{2} L(p-1) = mN - \frac{m(m-1)}{2p} L(p). \quad (7)$$

Since p and N are fixed, then (7) is increasing if

$$N > \frac{2m-1}{2p} L(p).$$

Since $m \leq p$ and $N > \left(p - \frac{1}{2}\right) L(p-1) = \frac{2p-1}{2p} L(p)$, then (7) is increasing. That means that its maximal sum comes when $m = p$. Therefore, for $S(N) \neq p$ we would need the sum with $m \leq q$ elements to surpass it. However, it would not be with less than q elements because S is a monotone increasing function (by Lemma 3), and $S(qL(q)) = q$, and $(p-1)L(p-1) + 1 > qL(q)$. Therefore, we need only check that the sum with q elements is not larger. With q elements we get

$$qN - \frac{q(q-1)}{2} L(q-1) = qN - \frac{q-1}{2} L(q) = qN - \frac{q-1}{2p} L(p). \tag{8}$$

Since $N > \frac{L(p)(p^2-p-q+1)}{2pk}$, the sum in (7) with $m = p$ is larger than (8). Therefore $S(N) = p$.

For the second part we have $m(N) \geq q + 1$. Take an N in the interval

$$\left(\left(p - \frac{1}{2}\right) L(p-1), \frac{L(p)(p^2 - p - q + 1)}{2pk}\right);$$

then the expression in (8) is greater than the expression in (7) for $m = p$. That implies that $S(N) \leq q$ for those values. But $S(qL(q)) = q$ and S is monotone increasing, so $S(N) = q$ for the whole interval. Therefore, $S(N) \neq M(N)$ for any N in the interval

$$\left(qL(q) + 1, \frac{L(p)(p^2 - p - q + 1)}{2pk}\right). \qquad \square$$

Now we are ready to prove our theorem about densities.

Proof of Theorem 5. For the upper density, let k be a positive integer. For almost all primes p, the closest prime power to p is a prime q with $p - q \geq k$. This is because the average distance between prime powers less than or equal to x is $\log x$ by the Prime Number Theorem, and most prime powers are primes. Let B_k be the set of such primes. For these primes p we have that the interval

$$\left[\max\left\{\left(p - \frac{1}{2}\right) L(p-1), \frac{L(p)(p^2 - p - q + 1)}{2pk}\right\}, pL(p)\right]$$

is a subset of $A(pL(p))$ (by Lemma 5). Now,

$$\lim_{p \to \infty} \frac{\left(p - \frac{1}{2}\right) L(p-1)}{pL(p)} = \lim_{p \to \infty} \frac{p - \frac{1}{2}}{p^2} = 0, \tag{9}$$

and

$$\lim_{p \to \infty} \frac{L(p)(p^2 - p - q + 1)/(2pk)}{pL(p)} = \lim_{p \to \infty} \frac{p^2 - p - q + 1}{2p^2 k} = \frac{1}{2k}. \tag{10}$$

Therefore,

$$\lim_{\substack{p\to\infty\\ p\in B_k}} \frac{A(pL(p))}{pL(p)} \geq 1 - \frac{1}{2k}.$$

This shows that $\delta^+ \geq 1 - \frac{1}{2k}$ for any fixed integer k. Therefore, $\delta^+ = 1$.

For the lower density, let C_k be the set of primes p such that the largest prime power less than p is a prime q with $p - q \leq k$. For $p \in C_k$, by Lemma 5, all elements N in the interval

$$\left(\left(p - \frac{1}{2}\right) L(p-1), \frac{L(p)(p^2 - p - q + 1)}{2pk}\right) \tag{11}$$

do not satisfy $M(N) = S(N)$. Consider $A\left(\frac{p}{2k}L(p)\right)$. As $p \to \infty$ the ratio of the right endpoint of (11) to $\frac{p}{2k}L(p)$ goes to 1. We also have

$$\lim_{p\to\infty} \frac{(p - \frac{1}{2})L(p-1)}{\frac{p}{2k}L(p)} = \lim_{p\to\infty} \frac{2k}{p} = 0. \tag{12}$$

Therefore, if we have infinitely many primes p in C_k, then from (12) we get

$$\lim_{\substack{p\to\infty\\ p\in C_k}} \frac{A\left(\frac{p}{2k}L(p)\right)}{\frac{p}{2k}L(p)} = 0.$$

Therefore, if there is a k, for which C_k is infinite, then $\delta^- = 0$. Recent work from Zhang [8], Maynard [5], and the Polymath group [6] has worked to find the best k they can. In particular, now we know that there exist infinitely many pairs of primes p, q such that $p - q \leq 246$. The techniques of their proof imply that the number of primes $p \leq x$ for which there is a prime q with $p - q \leq 246$ is greater than or equal to $C\frac{x}{\log^{50} x}$ for a positive constant C. Since the number of prime powers that are not prime is bounded above by a constant times $\sqrt{x}$, that means there are infinitely many primes in C_{246}. This implies that

$$\delta^- = 0. \qquad \square$$

Remark 1. The proof that $\delta^- = 0$ does not require the impressive work on prime gaps of Zhang and others. One could prove it using the Prime Number Theorem. However, one has to be more careful describing what is the infinite set on which the limit goes to 0, because we would not be able fix an integer k to form C_k.

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