Beatty Sequences and the Prime Race

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Background Knowledge

- Beatty Sequences?

- Named after Samuel Beatty who wrote about them in 1926 (Amer. Math. Monthly, 1926)
- A sequence formed by flooring successive positive multiples of a positive **irrational number ϴ**
- i.e. { ⌊ϴ⌋, ⌊2ϴ⌋, ⌊3ϴ⌋, … , ⌊nϴ⌋ } where ⌊x⌋ is the floor function (Weisstein, Eric W. "Beatty Sequence." From *MathWorld*--A Wolfram Web Resource.http://mathworld.wolfram.com/BeattySequence.html)

Example:

- If ϴ = √2, Beatty Sequence → {⌊1.41⌋, ⌊2.82⌋, ⌊4.24⌋…} → {1,2,4…}

So what makes Beatty Sequences unique?

- Given two positive irrational numbers α and β such that

$$\frac{1}{\alpha} + \frac{1}{\beta} = 1,$$

The union of the corresponding Beatty sequences $\lfloor\alpha\rfloor$, $\lfloor 2\alpha\rfloor$,… and $\lfloor\beta\rfloor$, $\lfloor 2\beta\rfloor$,… contain all positive integers without repetition.

Example:

- If $\alpha = \sqrt{2}$, $\beta = 2 + \sqrt{2}$
- Set α = {1,2,4,5,7,8,9,…} and Set β = {3,6,10,…}
- Set α ∪ β = {1,2,3,4,5,6,7,8,9,10…} and Set α ∩ β = { }

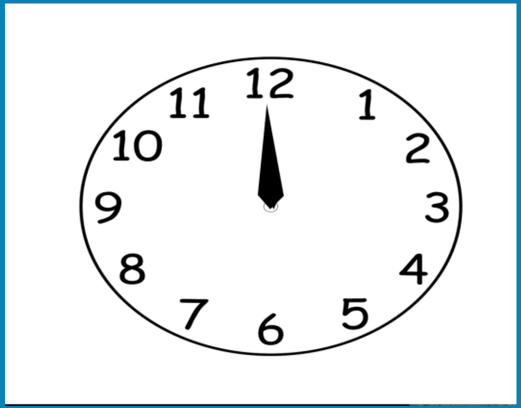
Modular arithmetic

- System of wrapping integers around a certain value (the modulus) using the "mod" function
- Imagine two whole integers a and q such that $\frac{a_n}{q} = k_n\ remainder\ r_n$.

e.g. $\frac{11}{5} = 2\ remainder\ 1$ and $\frac{19}{5} = 3\ remainder\ 4$

According to the above relation:

I.	$a_n = k_n q + r_n$	e.g. 11 = (5x2)+1
II.	$r < q\ (modulus)$	e.g. 1<5 and 4<5
III.	$a\ mod\ q = r$	→11mod5 =1 and 19mod5 = 4



Practice Example:

- It is currently 2:00pm on my analogue watch. What time will it be 13 hours from now?
  - 2+13 = 15 (analogue) ≡ 14+13 = 27 (digital)
  - 15 mod 12 = 3.00am (analogue) ≡ 27 mod 24 = 03:00 (digital)

Prime number theorem for arithmetic progressions and the Chebyshev Bias

- **Prime number (p):** Positive integer which can only be fully divided (leaving no remainder) by 1 and itself.

i.e. $\frac{p}{1} = k_1\ remainder\ 0$ and $\frac{p}{p} = k_2\ remainder\ 0$, where K is an integer and variable dividend.

Examples include: {2,3,5,7,11…}

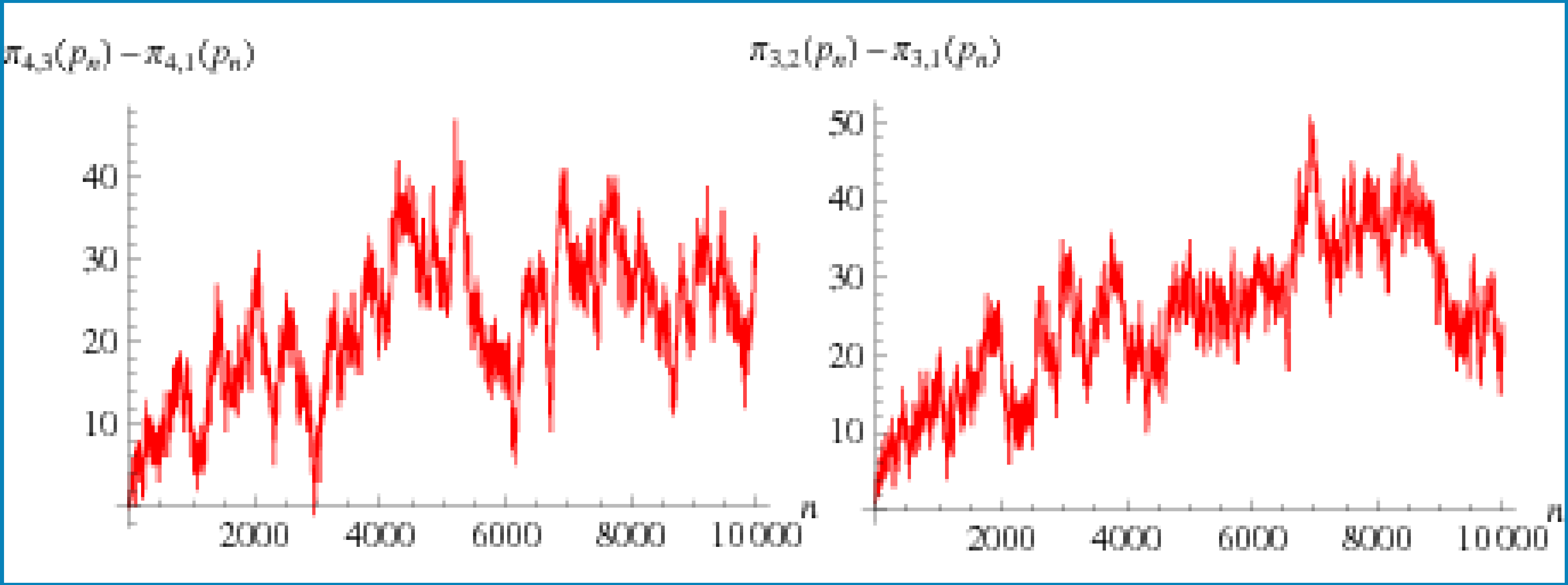
- **Arithmetic progressions:** Sequence of numbers with a constant difference **d** between successive terms e.g. {1,3,5,7,9…} where constant difference d =2.

- **Theorem:** "primes tend to be equally split amongst the various forms qn+a with gcd(a,q) =1 for any given modulus q. More precisely, we know that for two such eligible values a and b,

$$\frac{\#\{\text{primes } qn+a \leq x\}}{\#\{\text{primes } qn+b \leq x\}} \to 1 \ (\text{as } x \to \infty)$$

("Prime Number Races", 2006)

- Chebyshev Bias:
  - Despite the prime number theorem for arithmetic progressions, "Chebyshev noticed that the remainder upon dividing the primes by 4 gives 3 more often than 1. Similarly, dividing the primes by 3 gives 2 more often than 1."
  - i.e. Consider a list of the first n prime numbers mod 4: $\{2 mod 4, 3 mod 4, 5 mod 4, \ldots, p_n mod 4\} \rightarrow \{2,3,1, \ldots p_n mod 4\}$ In the above list, $\pi_{4,3}(p_n) > \pi_{4,1}(p_n)$



Weisstein, Eric W. "Chebyshev Bias." From *MathWorld*--A Wolfram Web Resource. http://mathworld.wolfram.com/ChebyshevBias.html

Research Question(s)

- How does the distribution of primes between complementary Beatty sequences (α and corresponding β) behave?

  I. After normalization, do more primes usually fall on one side (α or β)?

  II. Do consecutive primes within Beatty sequences repel each other?

  III. Is there a Chebyshev Bias within the Beatty sequences?

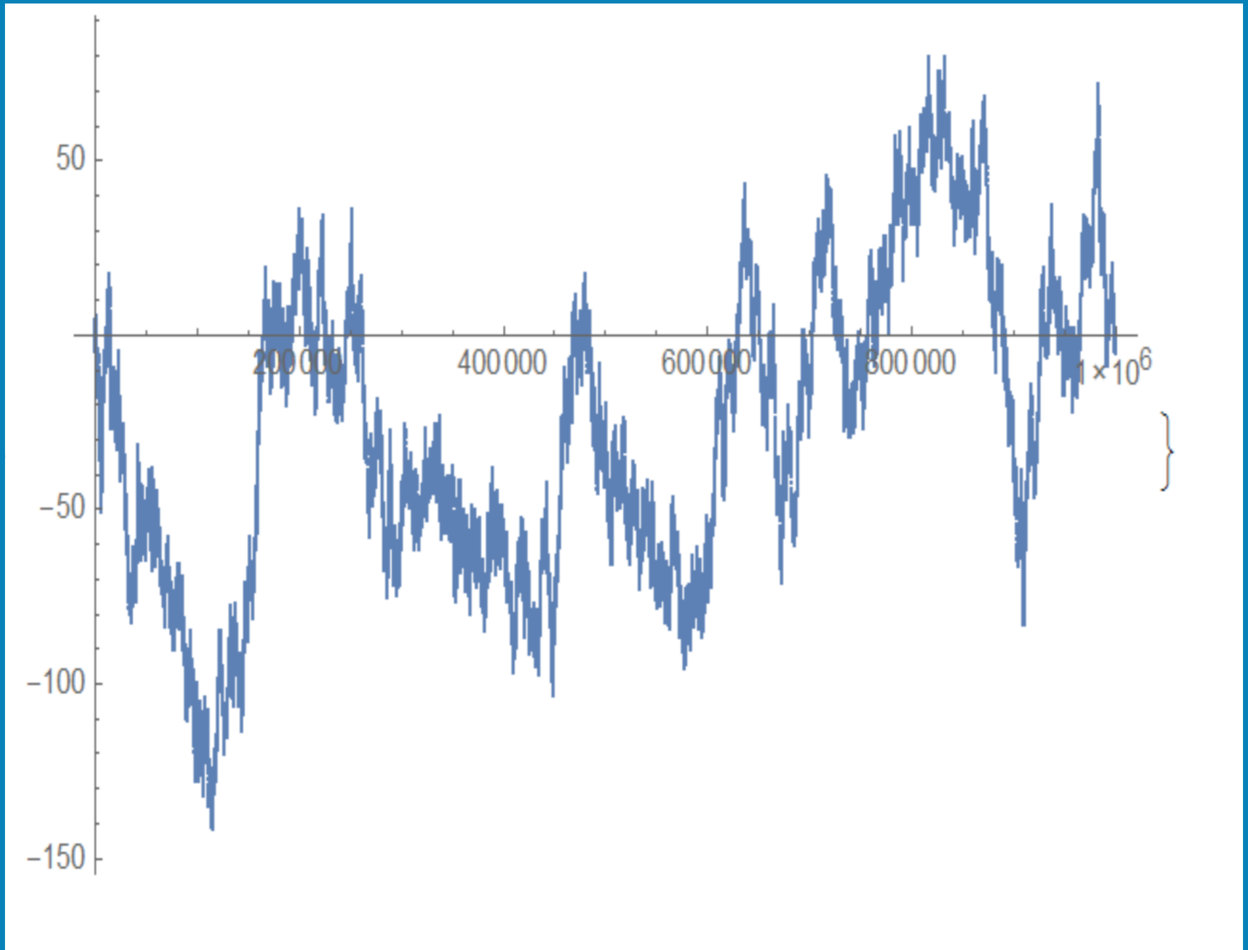
Research Methods

Section I of research question

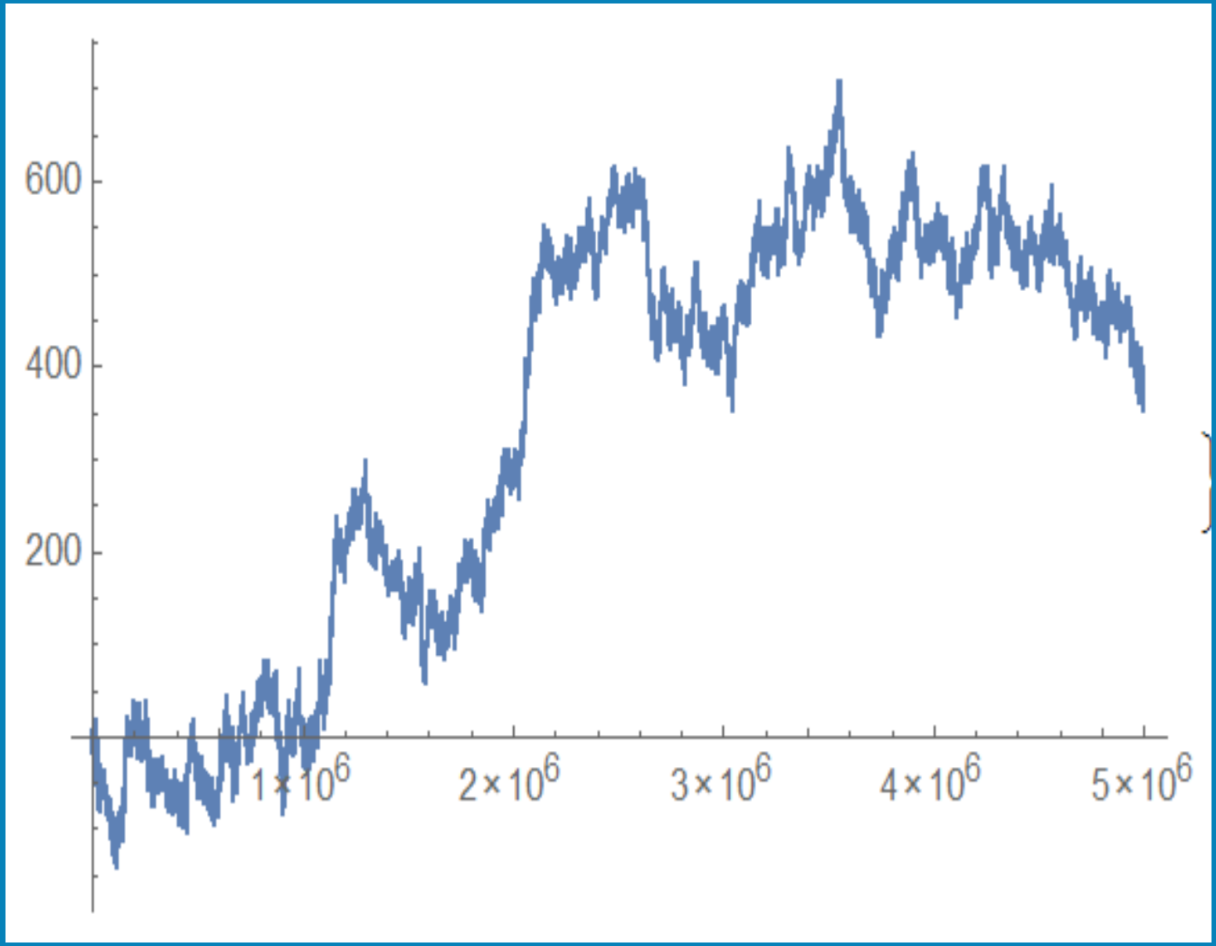
1. Created a function that derives pairs of complementary Beatty sequences (corresponding to α and its complement β) when only given the variable irrational number α.
2. Created a pair of functions ("Beatty1" and "Beatty2") each of which identified and counted the number of primes in each Beatty sequence corresponding to variables α and β respectively.
3. Created a function that normalized Beatty1 and Beatty2 counts
4. Ran above functions for different values of α and set ranges
5. Ran scatter plots representing the difference between Beatty1 and Beatty2 (Beatty1-Beatty2) so as to analyze distribution of primes

Results (Section I)

(Beatty1-Beatty2), $\alpha = \sqrt{2}$, Range: {n,1,1000000}

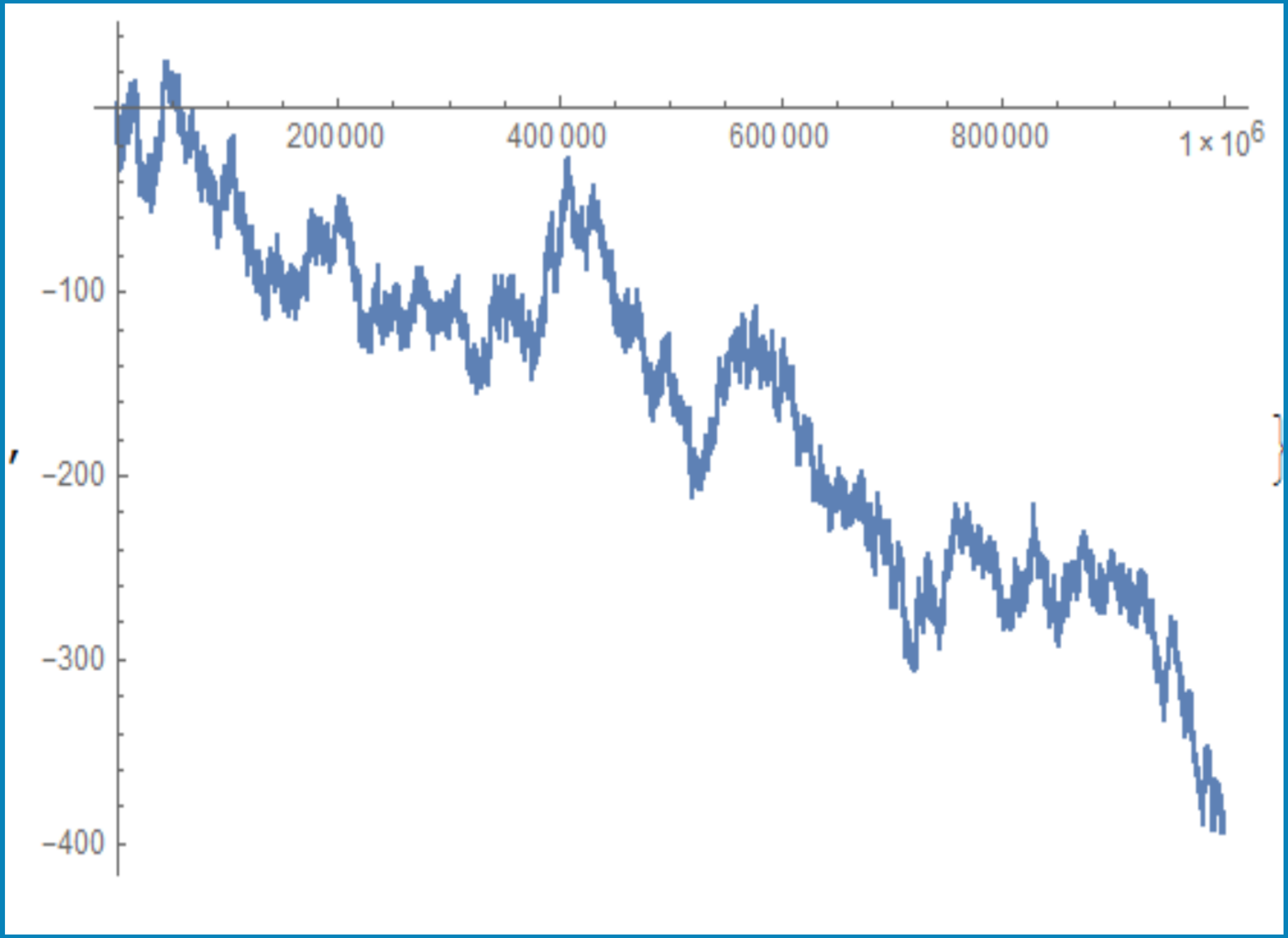


(Beatty1-Beatty2), $\alpha = \sqrt{2}$, Range: {n,1,5000000}

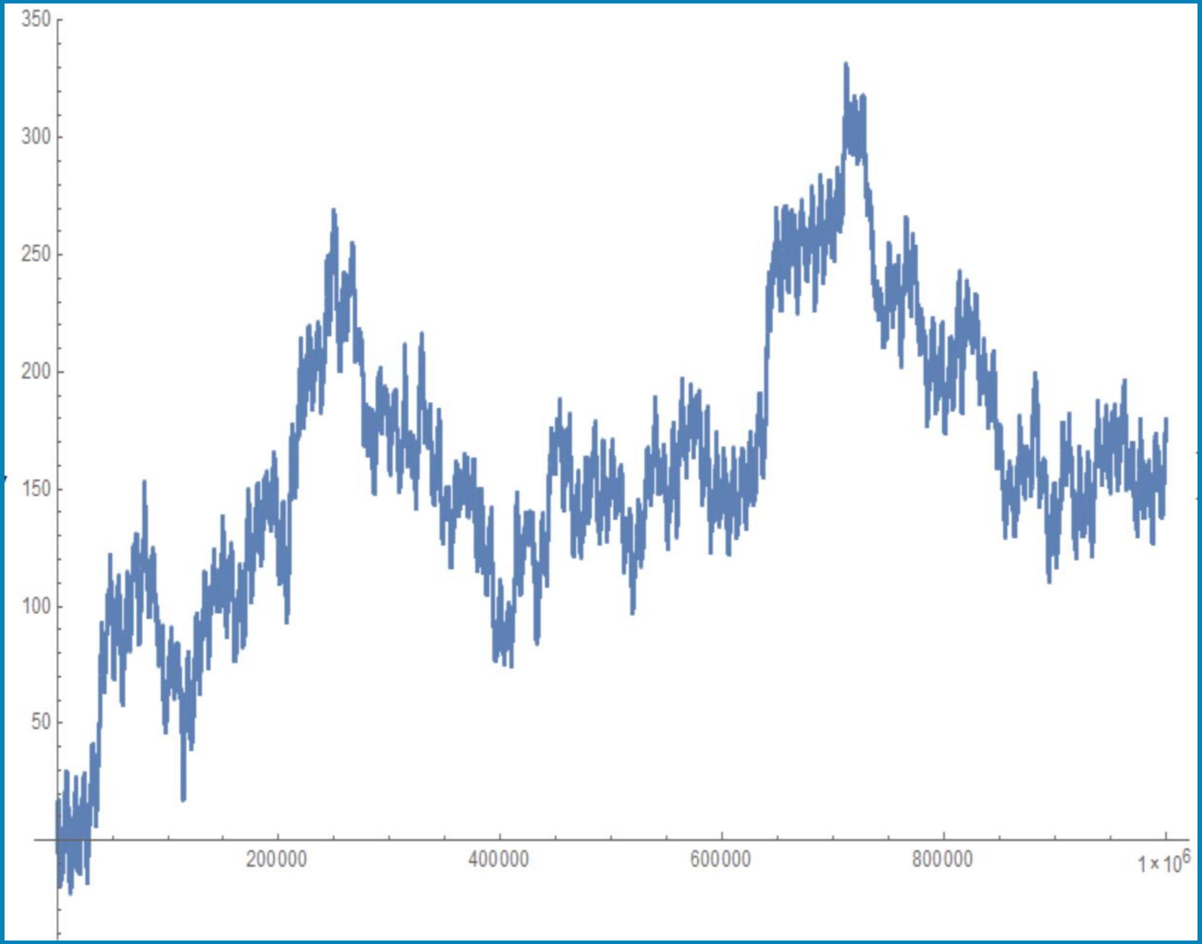


Results (Section I)

(Beatty1-Beatty2), $\alpha = \sqrt{3}$, Range: {n,1,1000000}



(Beatty1-Beatty2), $\alpha = \pi$, Range: {n,1,1000000}



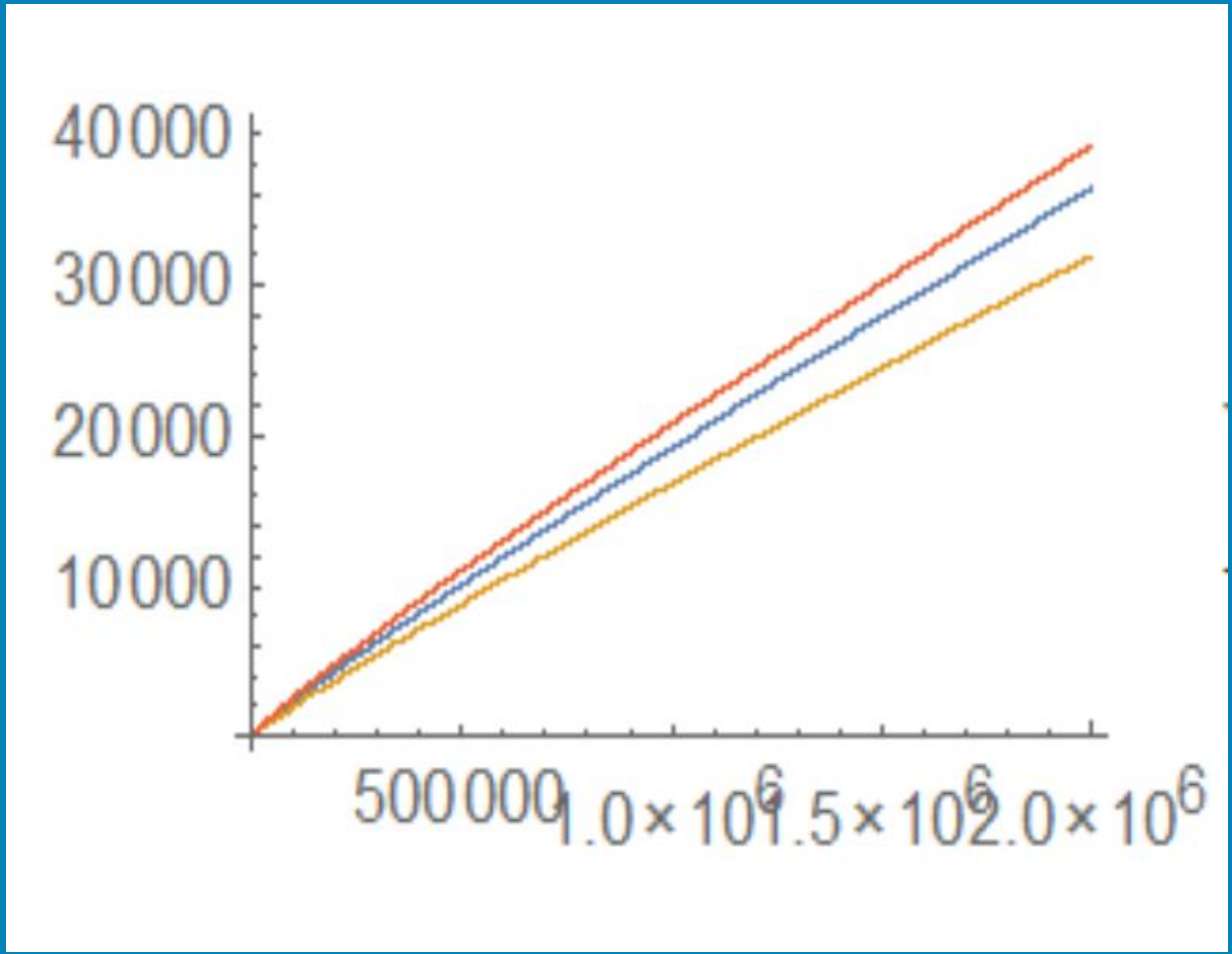
Research Methods

<u>Section II of research question</u>

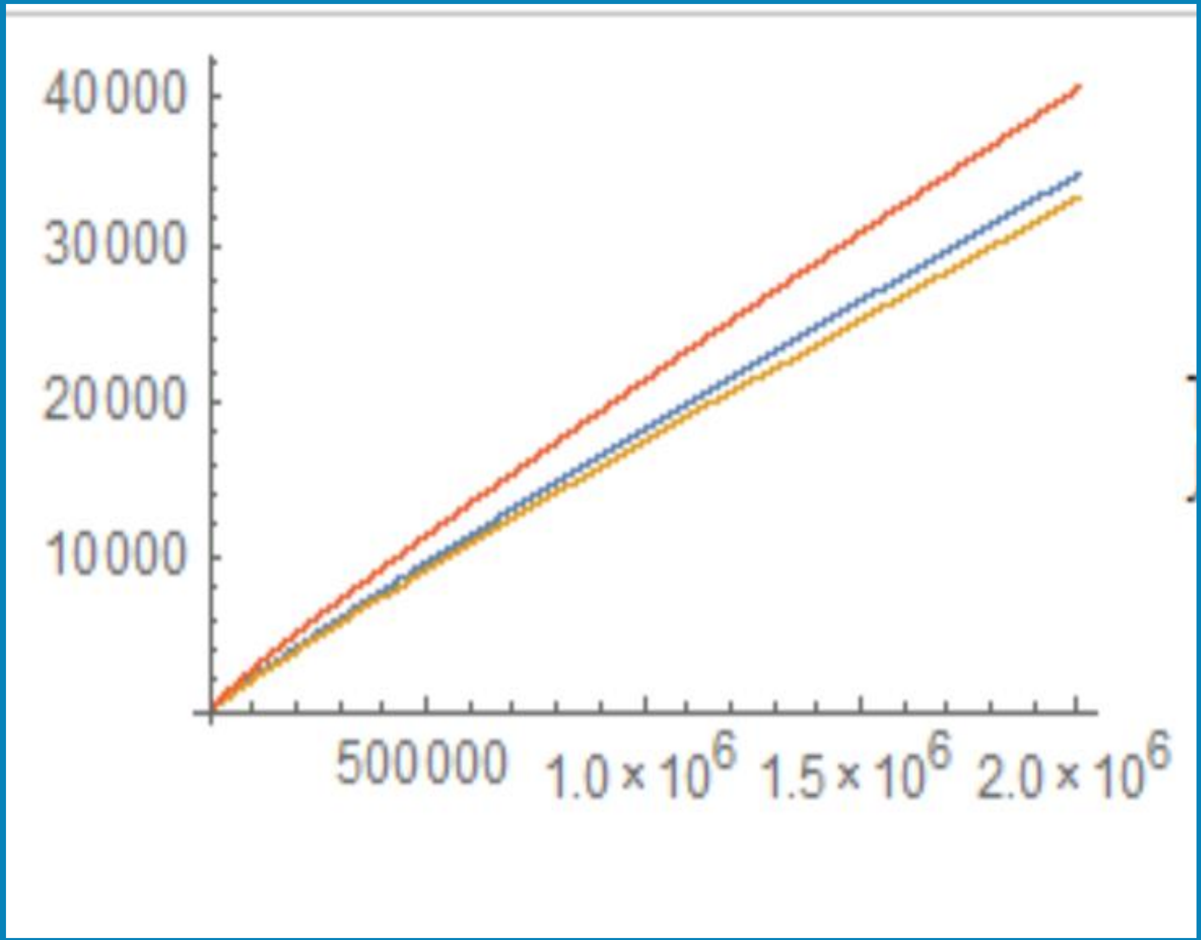
1. Created a function “AA” which identified prime numbers within Beatty Sequences corresponding to α, and measured the frequency at which consecutive prime numbers were within the sequence.
2. Above function (“AA”) was designed in such a way that it also measured the frequency at which consecutive prime numbers were within the sequence corresponding to β. These results were stored as the variable “BB.”
3. Created a function “AB” which identified prime numbers within the Beatty sequences corresponding to β, and measured the frequency at which the preceding prime number was within the sequence corresponding to α.
4. Above function (“AB”) was designed in such a way that it also measured the frequency at which prime numbers within the Beatty sequence corresponding to α were preceded by prime numbers within the sequence corresponding to β. These results were stored as “BA.”
5. Both functions (“AA” and “AB”) were normalized and ran for different values of α and β, as well as different ranges after which results were plotted to analyze the repulsion patterns.

Results (Section II)

α = √2, Range: {n,1,2000000}



α = √3, Range: {n,1,2000000}



Key:

Normalized BA

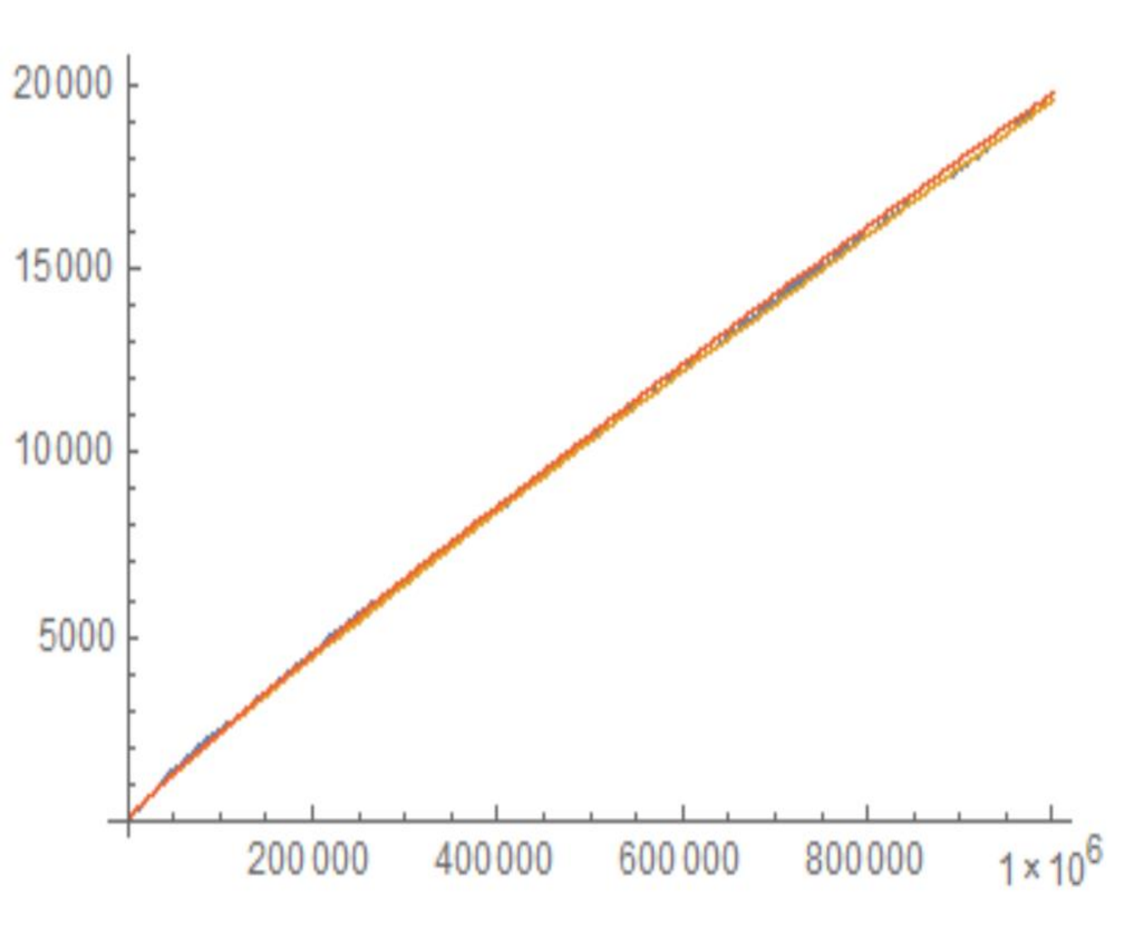
Normalized AA

Normalized BB

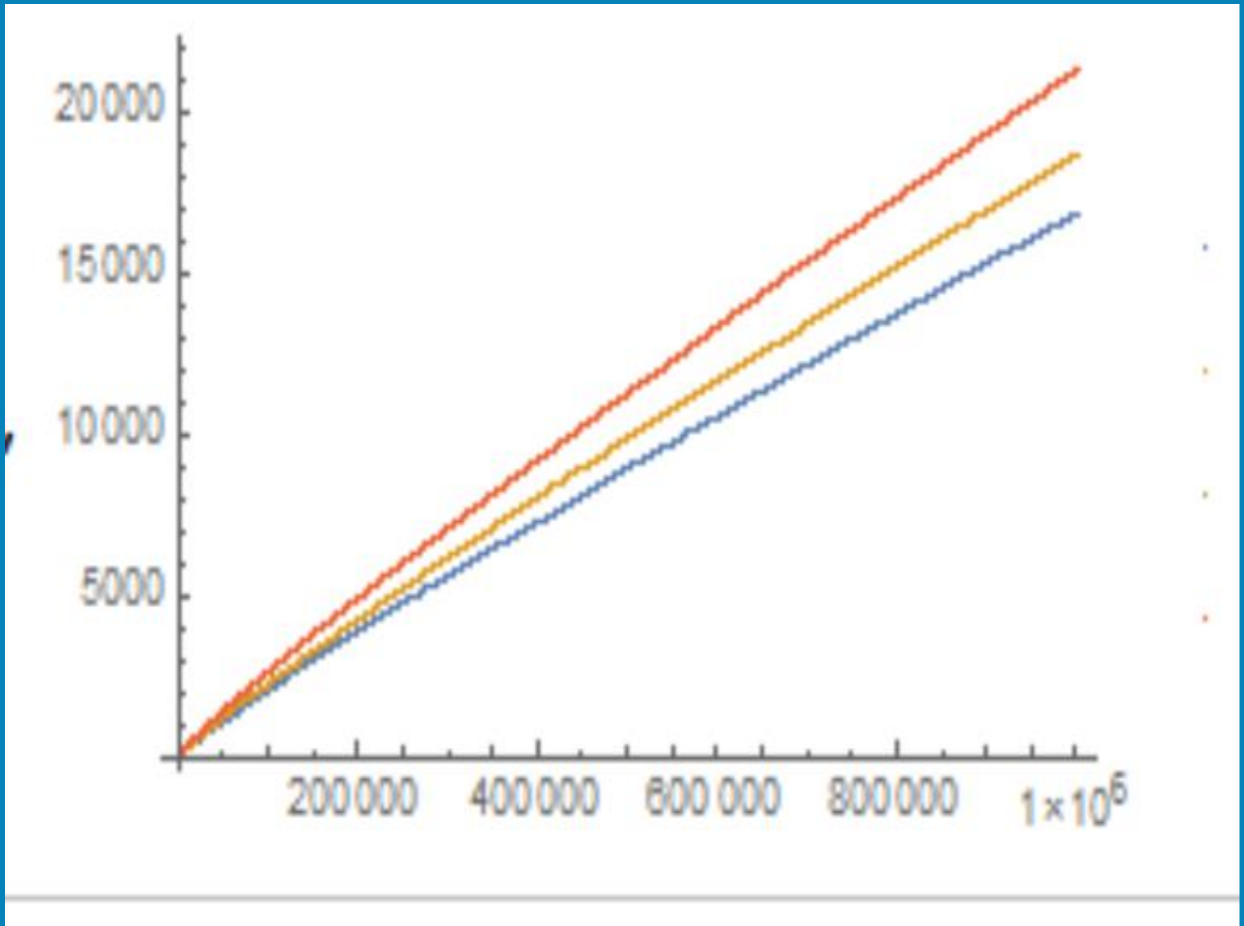
Normalized AB

Results (Section II)

α = π, Range: {n,1,1000000}



α = e, Range: {n,1,1000000}



Key:

Normalized BA

Normalized AA

Normalized BB

Normalized AB

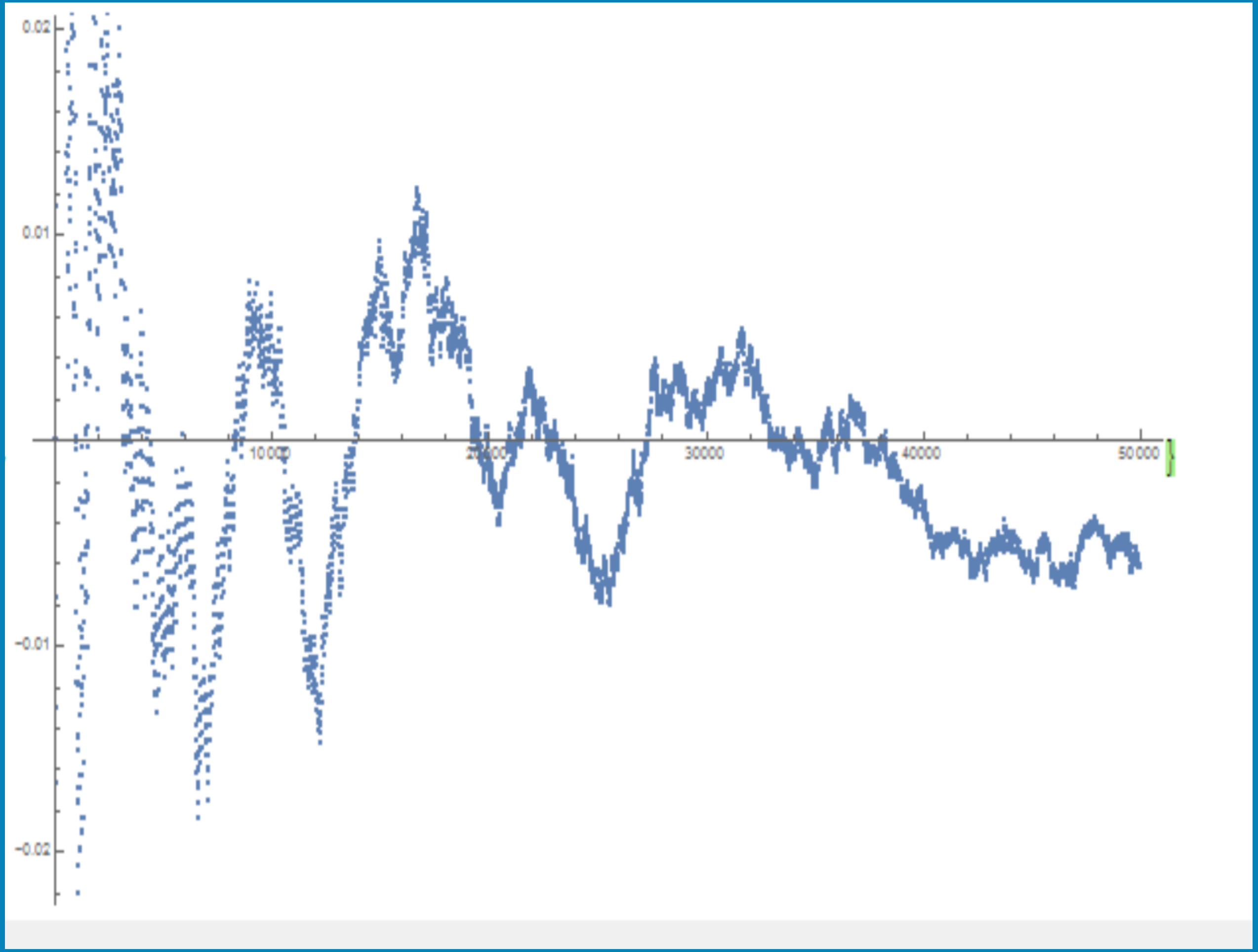
Research Methods

<u>Section III of research question</u>

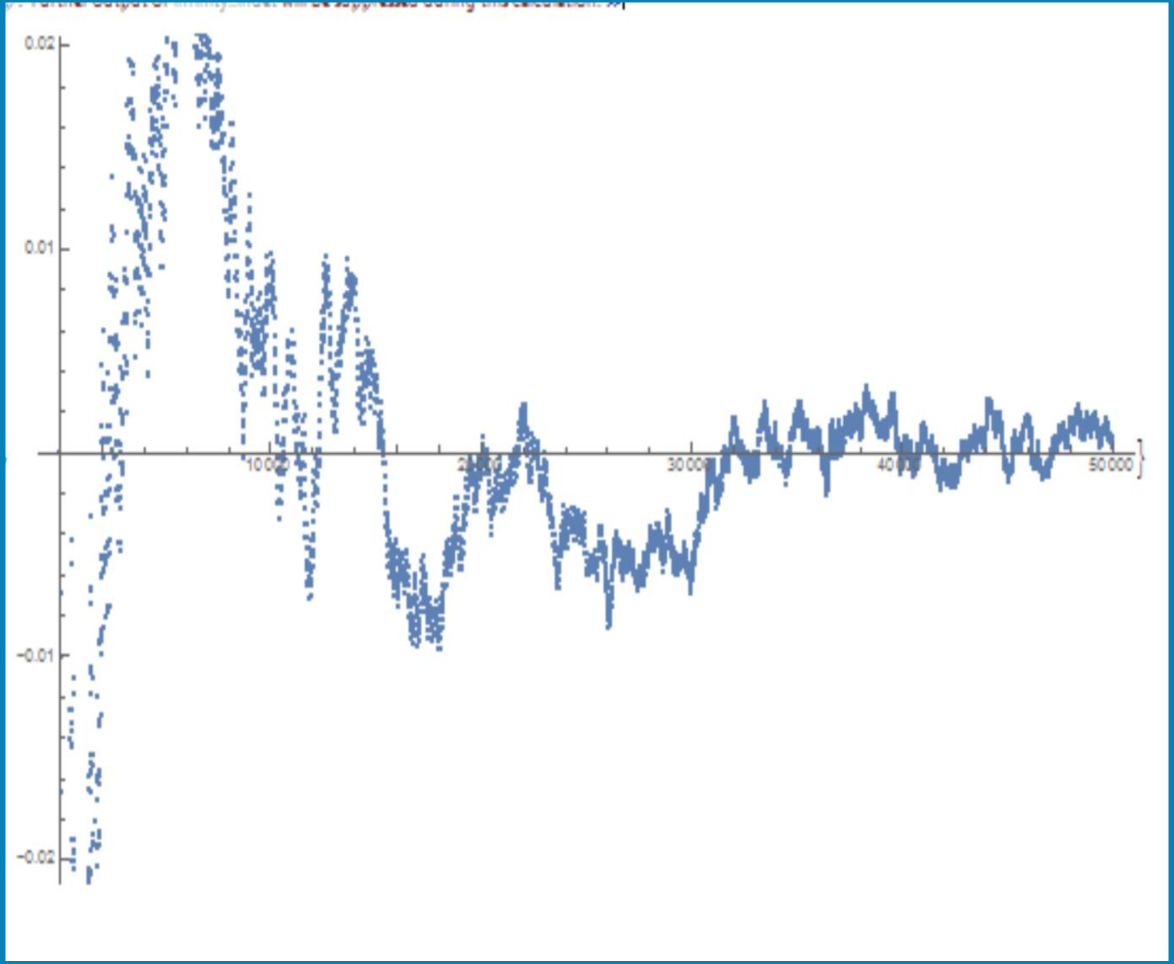
1. Created a function “BeattyP1” that identified and evaluated the number of primes congruent to 1mod3 within variable Beatty sequences.
2. Created a similar function “BeattyP2” that identified and evaluated the number of primes congruent to 2mod3 within variable Beatty sequences.
3. Normalized both functions above, and created a dependent function “NormalizedBeattyDif” which evaluated the difference: BeattyP2-BeattyP1, for different Beatty sequences.
4. The above process was repeated for a different modulus (mod4), whereby number of primes congruent to 1mod4 and 3mod4 were evaluated for different Beatty sequences.
5. The functions were ran for different irrational numbers (to derive various Beatty sequences) and different set ranges, after which the results were plotted for analysis.

Results (Section III)

NormalizedBeattyDif, mod3, $\alpha = \sqrt{2}$, Range: {n,1,50000}

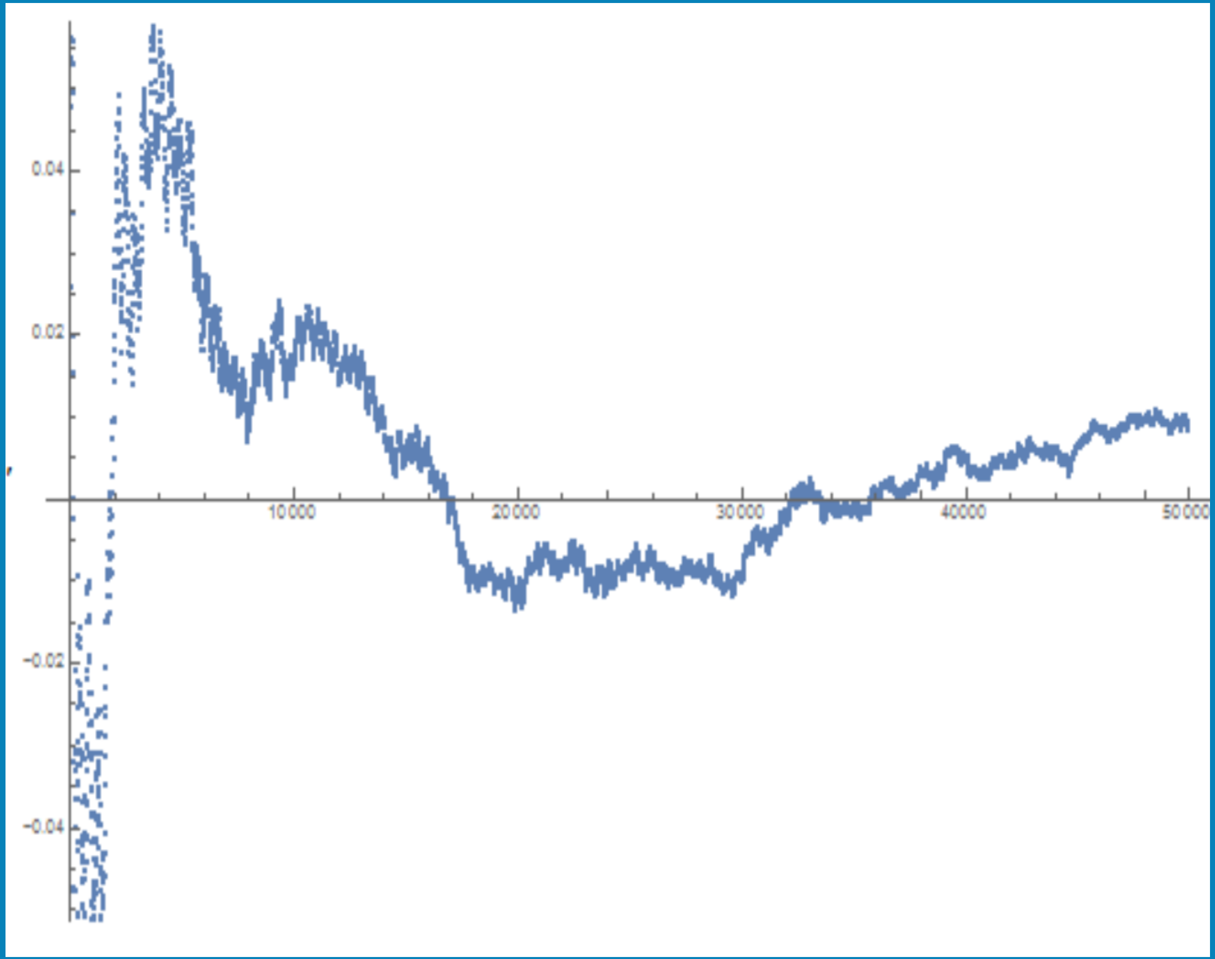


NormalizedBeattyDif, mod3, $\alpha = \sqrt{3}$, Range: {n,1,50000}

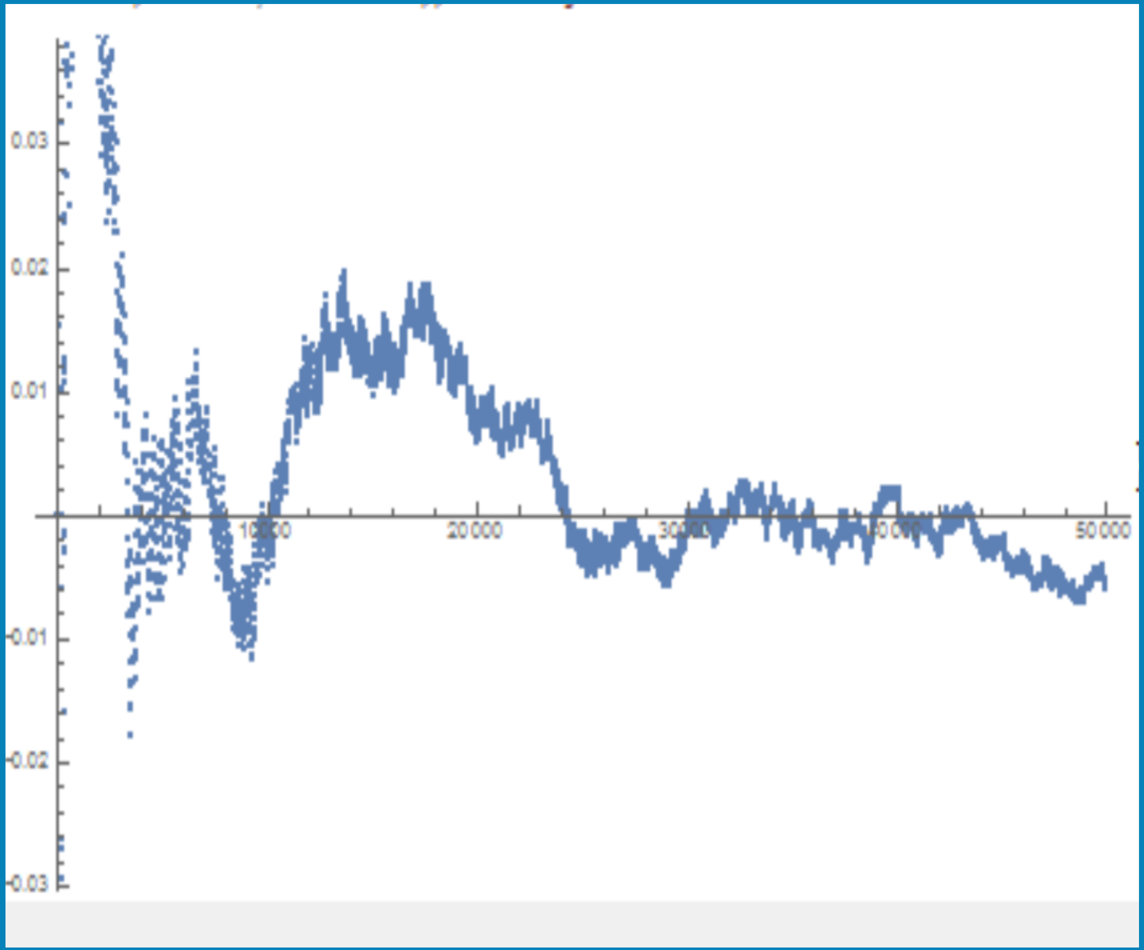


Results (Section III)

NormalizedBeattyDif, mod4, α = √2, Range: {n,1,50000}



NormalizedBeattyDif, mod4, α = √3, Range: {n,1,50000}



Conclusions

- Section I:
  - After normalization, both Beatty sequences (α and β) occasionally lead in the prime race but the consistency is limited to the set ranges and could thus simply be another phenomenon in line with "The Law of Small Numbers." (Prime Number Races, 2006)
- Section II:
  - Primes within the Beatty sequences examined tend to repel each other at a higher frequency relative to the frequency of attraction.
  - Frequency of attraction within set corresponding to α tends to be higher than within the set corresponding to β (except for transcendental number e).
  - The variance in repulsion frequencies within each sequence highlights the presence of a bias (except within the sequence corresponding to transcendental number π). However, the variance is not large enough to reject the results being consistent with randomness.
- Section III:
  - The Prime race between teams: 1mod3, 2mod3, 1mod4 and 3mod4 within the Beatty sequences examined are consistent with the pattern noted in the Chebyshev Bias.

Notes for future research:

- What is the nature of the Prime race within Beatty sequences that have larger scales? (n > 5×10^6)
- Do Beatty sequences remain consistent with the Chebyshev Bias at different moduli (other than 3 and 4)?
- Is the variance between Prime repulsion frequencies minimized within Beatty sequences that correspond to transcendental numbers?

Sources Cited

- Granville, Andrew, and Greg Martin. "Prime number races." *The American Mathematical Monthly* 113.1 (2006): 1-33.
- Sam, Beatty. “Problem 3173.” The American Mathematical Monthly (1926): 159.
- Weisstein, Eric W. "Beatty Sequence." From *MathWorld*--A Wolfram Web Resource.http://mathworld.wolfram.com/BeattySequence.html