Probabilistic Proof that $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$.

Let X be the sum of two n-sided dice. For $k = 2, 3, \ldots, n+1$, the probability that $X = k$ is $\frac{(k-1)}{n^2}$ because there are $k-1$ ways of adding two positive integers to k, namely $1 + (k-1), 2 + (k-2), \ldots, (k-1) + 1$. For $k = n + i$ with $i = 2, 3, \ldots, n$, the probability that $X = k$ is $\frac{n-i+1}{n^2}$ because there are $n - i + 1$ ways of adding up to k with two numbers from $\{1, 2, \ldots, n\}$, namely $i + n, (i+1) + (n-1), \ldots, n + i$. Since $2 \le X \le 2n$, then

$$1 = \sum_{k=2}^{2n} \mathbb{P}[X = k] = \sum_{k=2}^{n+1} \frac{k-1}{n^2} + \sum_{i=2}^{n} \frac{n-i+1}{n^2}$$

$$1 = \left(\frac{1}{n^2} + \frac{2}{n^2} + \cdots + \frac{n}{n^2}\right) + \left(\frac{n-1}{n^2} + \frac{n-2}{n^2} + \cdots + \frac{1}{n^2}\right)$$

$$n^2 = (1 + 2 + \cdots + n) + (1 + 2 + \cdots + (n-1))$$

$$n^2 = 2(1 + 2 + \cdots + n) - n.$$

Hence,

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

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