Sums of Proper Powers

We say that m is a *proper power* if $m = a^b$ for some integers $a, b > 1$. In [**2**], the authors proved that every positive integer $n \geq 33^{17} + 12$ can be written as the sum of four proper powers. We improve their result to:

Theorem. *All $n \geq 28$ can be represented as a sum of four proper powers. Furthermore, the only exceptions are*

$$1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ 10,\ 11,\ 12,\ 13,\ 14,\ 15,\ 17,\ 18,\ 19,\ 22,\ 23,\ 27.$$

Proof. Let $n \geq 33^3 + 12$ be an odd number. Since $\gcd(\phi(32), 3) = 1$ (where ϕ is the Euler totient function), a^3 runs through all odd residue classes modulo 32 as a runs through all odd residues modulo 32. Let $P \in \{3^3, 5^3, \ldots, 33^3\}$ be a cube such that $P \equiv n - 12 \bmod 32$. Then $n - P = 4(8k + 3)$. Gauss showed that every positive integer not congruent to $0, 4, 7$ mod 8 is a sum of three coprime squares [**1**, p. 262]. Hence we may choose integers x, y, z with $x^2 + y^2 + z^2 = 8k + 3$. Looking mod 8 we see x, y, z are all odd, so we may assume $x, y, z \geq 1$. But then $n = P + (2x)^2 + (2y)^2 + (2z)^2$. Therefore, for all odd $n \geq 33^3 + 12 = 35949$, we have a representation of n as a sum of four proper powers.

Let $n \geq 49^3 + 45 = 117693$ be even. Since $\gcd(3, \phi(48)) = 1$, as a runs through all odd residues modulo 48, a^3 also runs through all the odd residues modulo 48. Therefore, we can find an odd cube $P \in \{3^3, 5^3, \ldots, 49^3\}$ such that $P \equiv n - 45 \bmod 48$. Hence $n = P + (48k + 45)$ for some integer k. Since $48k + 45 \equiv 5 \bmod 8$, there exist integers $x, y, z \geq 0$ such that $x^2 + y^2 + z^2 = 48k + 45$ with $\gcd(x, y, z) = 1$. We want to show that none of x, y, z is 0 or 1. First, note that $x^2 + y^2 \equiv 0$ mod 3 implies $x \equiv y \equiv 0 \bmod 3$. If also $z = 0$, then $\gcd(x, y, z) \geq 3$. Therefore $x, y, z \geq 1$. If one of x, y, z is 1, then (after relabeling) $x^2 + y^2 = 48k + 44 \equiv 12 \bmod 16$. The squares modulo 16 are $0, 1, 4, 9$. We cannot add two of them to get $12 \bmod 16$; therefore the representation of $48k + 45$ as $x^2 + y^2 + z^2$ includes only proper powers. Hence, $n = P + x^2 + y^2 + z^2$ is the sum of four proper powers. Therefore, for $n \geq 49^3 + 45$ we have a representation of n as a sum of four proper powers.

To conclude one needs to check $n < 117693$. We computed all possible sums of four proper powers up to 117693 and we found all numbers between 1 and 117693 that are not represented as a sum of four proper powers. ■

REFERENCES

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2. Schinzel, A., Sierpiński, W. (1965). Sur les puissances propres. *Bull. Soc. Roy. Sci. Liège.* 34: 550–554.

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