Prime gaps: a breakthrough in number theory

Enrique Treviño

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LAKE FOREST COLLEGE

Introduction

The first 10 prime numbers are:
2, 3, 5, 7, 11, 13, 17, 19, 23, 29.
The gaps between them are:
1, 2, 2, 4, 2, 4, 2, 4, 6.
A few questions about gaps:

- Can we have arbitrarily large gaps between two consecutive primes?
- What is the average gap?
- If we consider all primes less than a bound (let's call it x) what is the biggest gap?
- Does 2 appear infinitely often as a gap between two primes?

Prime Gaps

- **Can we have arbitrarily large gaps between two consecutive primes?**
  Yes,
  $n! + 2, n! + 3, \ldots n! + n$ are all composites for any $n \geq 2$, so there is at least a gap of $n! + (n+1) - (n! + 1) = n$ between two consecutive primes.
- **If we consider all primes less than x, what is the average gap?**
  The prime number theorem implies that the k-th prime is approximately $k \log k$, from that it easy to show that the average gap is $\log x$.

Large Gaps

If we consider all primes less than a bound (let's call it x) what is the biggest gap?

Consider the proof using $n! + 2, n! + 3, \ldots$. This shows a gap of n, but between numbers of size $n!$. This proof would only yield a gap of the order $\log x / \log\log x$. This doesn't even match the average gap!

Erdős proved that you can beat the average gap by proving the largest gap is at least

$$C \log x \frac{\log\log x}{(\log\log\log x)^2},$$

for some constant C. In 1963 Rankin improved this to

$$c_0 \log x \frac{\log\log x \log\log\log\log x}{(\log\log\log x)^2}.$$

Twin Prime Conjecture

Does 2 appear infinitely often as a gap between two primes?

The twin prime conjecture conjectures that the answer to this is yes. In fact using a probabilistic heuristic we can even predict how many twin prime pairs we should have up to x. The conjecture is that there are about $C_2 x/(\log x)^2$ for a special constant C_2 called the twin prime constant.

We can't prove the twin prime conjecture, but can we say anything about "short gaps" between primes?

Twin Prime Conjecture

Our main question here is whether 2 appears infinitely often or not as a gap between primes. Note that this is equivalent to:

Conjecture (Twin Prime Conjecture)

Let $d_n = p_{n+1} - p_n$ where p_n is the n-th prime number. Then

$$\liminf_{n\to\infty} d_n = 2.$$

Note that by the prime number theorem:

$$\liminf_{n\to\infty} \frac{d_n}{\log p_n} \leq 1.$$

Results on short gaps

- $\liminf_{n\to\infty} \frac{d_n}{\log p_n} \leq 1.$
- $\liminf_{n\to\infty} \frac{d_n}{\log p_n} \leq 1 - c$, for some $c > 0$ (Erdős, 1940).
- $\liminf_{n\to\infty} \frac{d_n}{\log p_n} \leq \frac{1}{2}$ (Bombieri-Vinogradov, 1966).
- $\liminf_{n\to\infty} \frac{d_n}{\log p_n} \leq \frac{1}{2}e^{-\gamma} = 0.2807\ldots$ (Maier 1988).
- $\liminf_{n\to\infty} \frac{d_n}{\log p_n} \leq 1/4$ (Maier).
- $\liminf_{n\to\infty} \frac{d_n}{\log p_n} = 0. (Goldston - Pintz - Yildirim, 2005)$

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Goldston-Pintz-Yildirim

Theorem (GPY)

Let $\epsilon > 0$,

$$\liminf_{n\to\infty} \frac{d_n}{(\log p_n)^{1/2+\epsilon}} = 0.$$

Furthermore, if the Elliott-Halberstam conjecture is true

$$\liminf_{n\to\infty} d_n \leq 16.$$

Bombieri-Vinogradov

Let

$$\theta(x; q, m) = \sum_{\substack{p \le x \\ p \equiv m \bmod q}} \log p.$$

Theorem (Bombieri-Vinogradov)

Let $A > 0$ be fixed. There exist constants $B = B(A)$ and $c = c(A)$ such that

$$\sum_{q \le Q} \max_{m \bmod q} \left| \theta(x; q, m) - \frac{x}{\phi(q)} \right| \le c \frac{x}{(\log x)^A},$$

for $Q = \dfrac{\sqrt{x}}{\log x^A}$.

Elliott-Halberstam

Let

$$\theta(x; q, m) = \sum_{\substack{p \leq x \\ p \equiv m \bmod q}} \log p.$$

Conjecture (Elliott-Halberstam)

For any fixed $A > 0$ *and* $0 < \eta < 1/2$*, There exists a constant* c *such that*

$$\sum_{q \leq Q} \max_{m \bmod q} \left| \theta(x; q, m) - \frac{x}{\phi(q)} \right| \leq c \frac{x}{(\log x)^A},$$

for $Q = x^{1/2+\eta}$.

GPY Main Theorem

The set $\{a_1, a_2, \ldots a_k\}$ of integers $a_1 < a_2 < \ldots a_k$ is said to be admissible if there is no prime p such that p divides $P(n) = (n + a_1)(n + a_2) \cdots (n + a_k)$ for all integers n.

Theorem

Let $k \geq 2, l \geq 1$ be integers and $0 < \eta < 1/2$ such that

$$1 + 2\eta > \left(1 + \frac{1}{2l+1}\right)\left(1 + \frac{2l+1}{k}\right).$$

If the Elliott Halberstam conjecture is true for $Q = x^{1/2+\eta}$, then if $\{a_1, a_2, \ldots, a_k\}$ is admissible, then there are infinitely many integers n such that at least two of $n + a_1, n + a_2, \ldots, n + a_k$ are prime.

Zhang's Theorem

Theorem (Zhang, May 14 2013)

Let $A > 0$. There exist constants eta, $\delta, c > 0$ such that for any given integer a, we have,

$$\sum_{\substack{q \le Q \\ (q,m)=1 \\ q \text{ is } y-\text{smooth} \\ q \text{ squarefree}}} \left| \theta(x; q, m) - \frac{x}{\phi(q)} \right| \le c \frac{x}{(\log x)^A},$$

where $Q = x^{1/2+\eta}$ and $y = x^{\delta}$.

Zhang managed to prove this with $\eta/2 = \delta = 1/1168$.

Zhang's Theorem

Consequences:

Theorem (Zhang)

Let $k \geq 3500000$. If $\{a_1, a_2, \ldots, a_k\}$ is an admissible set, then there are infinitely many n for which at least two of $n + a_1, n + a_2, \ldots, n + a_k$ are prime.

Corollary (Zhang)

$$\liminf_{n \to \infty} d_n \leq 70000000.$$

Polymath8 Progress

Date	ϖ or (ϖ, δ)	k_0	H
Aug 10 2005		6 [EH]	16 [EH] ([Goldston-Pintz-Yildirim])
May 14 2013	1/1,168 (Zhang)	3,500,000 (Zhang)	70,000,000 (Zhang)
May 21			63,374,611 (Lewko)
May 28			59,874,594 (Trudgian)
May 30			59,470,640 (Morrison) 58,885,998? (Tao) 59,093,364 (Morrison) 57,554,086 (Morrison)
May 31		2,947,442 (Morrison) 2,618,607 (Morrison)	48,112,378 (Morrison) 42,543,038 (Morrison) 42,342,946 (Morrison)
Jun 1			42,342,924 (Tao)
Jun 2		866,605 (Morrison)	13,008,612 (Morrison)

Polymath8 Progress

Jun 3	1/1,040? (v08ltu)	341,640 (Morrison)	4,982,086 (Morrison) 4,802,222 (Morrison)
Jun 4	1/224?? (v08ltu) 1/240?? (v08ltu)		4,801,744 (Sutherland) 4,788,240 (Sutherland)
Jun 5		34,429? (Paldi/v08ltu) 34,429? (Tao/v08ltu/Harcos)	4,725,021 (Elsholtz) 4,717,560 (Sutherland) 397,110? (Sutherland) 4,656,298 (Sutherland) 389,922 (Sutherland) 388,310 (Sutherland) 388,284 (Castryck) 388,248 (Sutherland) 388,188 (Sutherland) 387,982 (Castryck) 387,974 (Castryck)
Jun 6	~~(1/488,3/9272)~~ (Pintz) ~~1/552~~ (Pintz, Tao)	~~60,000*~~ (Pintz) ~~52,295*~~ (Peake) 11,123 (Tao)	387,960 (Angelveit) 387,910 (Sutherland) 387,904 (Angeltveit) 387,814 (Sutherland) 387,766 (Sutherland)

Polymath8 Progress

			~~768,534*~~ (Pintz)
Jun 7	~~(1/538, 1/660)~~ (v08ltu) ~~(1/538, 31/20444)~~ (v08ltu) ~~(1/942, 19/27004)~~ (v08ltu) $828\varpi + 172\delta < 1$ (v08ltu/Green)	~~11,018~~ (Tao) ~~10,721~~ (v08ltu) ~~10,719~~ (v08ltu) ~~25,111~~ (v08ltu) 26,024? (vo8ltu)	~~113,520~~? (Angeltveit) ~~109,314~~? (Angeltveit/Sutherland) ~~707,328*~~ (Sutherland) ~~108,990~~ (Sutherland) ~~113,462*~~ (Sutherland) ~~112,302*~~ (Sutherland) ~~112,272*~~ (Sutherland) ~~116,386*~~ (Sun) ~~108,978~~ (Sutherland) ~~108,634~~ (Sutherland) ~~108,632~~ (Castryck) ~~108,600~~ (Sutherland) ~~108,570~~ (Castryck) ~~108,556~~ (Sutherland) ~~108,550~~ (xfxie) ~~275,424~~ (Sutherland) ~~108,540~~ (Sutherland) ~~275,418~~ (Sutherland)

Polymath8 Progress

Jun 24	$(134+\frac{2}{3})\varpi + 28\delta \le 1$? (v08ltu) $140\varpi + 32\delta < 1$? (Tao) ~~1/88?? (Tao)~~ ~~1/74?? (Tao)~~	1,268? (v08ltu)	10,206? (Engelsma)
Jun 25	$116\varpi + 30\delta < 1$? (Fouvry-Kowalski-Michel-Nelson/Tao)	1,346? (Hannes) ~~502?? (Trevino)~~ 1,007? (Hannes)	10,876? (Engelsma) ~~3,612?? (Engelsma)~~ 7,860? (Engelsma)
Jun 26	$116\varpi + 25.5\delta < 1$? (Nielsen) $(112+\frac{4}{7})\varpi + (27+\frac{6}{7})\delta < 1$? (Tao)	962? (Hannes)	7,470? (Engelsma)
Jun 27	$108\varpi + 30\delta < 1$? (Tao)	902? (Hannes)	6,966? (Engelsma)
Jul 1	$(93+\frac{1}{3})\varpi + (26+\frac{2}{3})\delta < 1$? (Tao)	873? (Hannes) ~~872? (xfxie)~~	6,712? (Sutherland) ~~6,696? (Engelsma)~~
Jul 5	$(93+\frac{1}{3})\varpi + (26+\frac{2}{3})\delta < 1$ (Tao)	720 (xfxie/Harcos)	5,414 (Engelsma)
Jul 10	7/600? (Tao)		

Polymath8 Results

Polymath8 was able to get the following results:

- $\eta \leq 7/300$ (improving Zhang's result of $\eta \leq 1/584$)
- $k = 632$ (improving from 3500000)
- $\liminf_{n\to\infty} d_n \leq 4680$ (improving from 70 000 000).

Maynard and Polymath8b

In November of 2013, Maynard, a postdoc at U. Montreal came out with a different proof of the bounded small gaps. A proof that does not require an improvement on Bombieri-Vinogradov:

Theorem (Maynard)

$$\liminf_{n\to\infty} d_n \leq 600.$$

Furthermore if EH is true for any $\eta < 1/2$*, then*

$$\liminf_{n\to\infty} d_n \leq 12.$$

Polymath8b

Polymath8 joined Maynard and they are improving his result. The latest results (updated March 1, 2014) are:

Theorem (Polymath8b)

$$\liminf_{n\to\infty} d_n \leq 252.$$

Furthermore if EH is true for any $\eta < 1/2$, *then*

$$\liminf_{n\to\infty} d_n \leq 6.$$

The 252 might go down a bit more but the 6 is staying put. The famous sieve parity barrier is preventing any improvement there.