On a sequence related to the factoradic representation of an integer

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Abstract

For a positive integer r, define j_r to be the smallest positive integer n satisfying $n! > n^{r-1}$. In this paper we prove $j_{r+1} \in \{j_r + 1, j_r + 2\}$, which leads us to explore the set of positive integers r for which $j_{r+1} = j_r + 2$. We prove this set has the same density as the prime numbers. The sequence j_r was introduced by Carlson, Goedhart, and Harris in their work on factoradic happy numbers, and we prove some properties of j_r that lead to an improvement on a theorem of theirs.

1 Introduction

Let r be a positive integer and define j_r to be the smallest positive integer n satisfying

$$n! > n^{r-1}. \tag{1}$$

The first 20 values of j_r (A230319 in the On-line Encyclopedia of Integer Sequences [9]) are

$$\{2, 3, 4, 6, 7, 8, 10, 11, 12, 14, 15, 16, 18, 19, 20, 22, 23, 24, 25, 27\}. \tag{2}$$

The sequence j_r made an appearance in work by Carlson, Goedhart, and Harris [2] on *factoradic happy numbers*. Our main goal in this paper is to prove some properties of j_r to be able to improve some of their results.

To get a better background, we need to define *happy numbers* (A007770) and *factoradic* expansion. Let n be a positive integer and $S_2(n)$ be the sum of the squares of its decimal digits. Consider the sequence of iterates of S_2 on n, i.e., $n, S_2(n), S_2^2(n), \ldots$. It is well known [7, pp. 74, 83–84] that eventually all the terms in the sequence are 1 or eventually the sequence becomes periodic with the cycle

$$4 \to 16 \to 37 \to 58 \to 89 \to 145 \to 42 \to 20 \to 4.$$

If the sequence reaches 1, we say n is *happy*. Many generalizations of happy numbers have been studied. For example, one can allow to change the base to $b \geq 2$ instead of 10, and one can replace sum of squares of digits, with the sum of r-th powers of the digits for some integer $r \geq 1$. Let $S_{r,b}(n)$ be the sum of r-th powers of the digits of n when n is written in base b. In depth analysis of the cases $r \in \{2, 3\}$, $b \in \{2, 3, \ldots, 10\}$ has been done by Grundman and Teeple [4]. The techniques developed by Grundman and Teeple [4] can easily be used to study other choices of r and b. Another generalization is to allow the base b to be a negative number. This has been done by Grundman and Harris [6] for $-2 \geq b \geq -10$ and $r = 2$. The authors also study in what cases there exist consecutive b-happy numbers in an arithmetic progression, generalizing work of El-Sedy and Siksek [3] (who do this for happy numbers) and the work of Grundman and Teeple [5] (who do this for b-happy r-power happy numbers). Bland, Cramer, de Castro, Domini, Edgar, Johnson, Klee, Koblitz, and Sundaresan [1] addressed a series of questions regarding a generalization of happy numbers to the fractional base 3/2, and Treviño and Zhylinski [10] addressed other fractional bases.

Every positive integer n can be written uniquely in the form

$$n = \sum_{i=1}^{k} a_i \cdot i!,$$

for some positive integer k satisfying $1 \leq a_k \leq k$, and $0 \leq a_i \leq i$ for $1 \leq i \leq k-1$. We call this the factoradic expansion of n. We will use the notation $n = (a_k a_{k-1} \cdots a_1)_!$ to express a number written in its factoradic expansion. For example, $8 = 110_!$ because $8 = 0 \cdot 1! + 1 \cdot 2! + 1 \cdot 3!$. Carlson, Goedhart, and Harris [2] generalized the concept of happy numbers to factoradic expansions as follows: let $S_{r,!}(n)$ be the sum of the r-th powers of the factoradic digits of a number n, then a positive number n is an r-power factoradic happy number if there exists an integer k such that $S_{r,!}^k(n) = 1$ (the k-iteration of $S_{r,!}$ is 1). Their main theorem is that for $r \in \{1, 2, 3, 4\}$, there exist arbitrarily long sequences of consecutive r-power factoradic happy numbers.

When studying happy numbers, regardless of the setting, it is important to show that the relevant happy function S ($S = S_{r,b}$ or $S = S_{r,!}$) satisfies $S(n) < n$ for all $n > N$, for some integer N. One of the difficulties of the factoradic case is the bound for N is less easy to get that in the other happy number generalizations. Carlson, Goedhart, and Harris [2, Theorem 10] proved that for $2 \leq r \leq 30$, they can choose $N = M_r = \sum_{i=1}^{j_r} i \cdot i! = (j_r + 1)! - 1$. Our main motivation to study j_r is to be able to prove this result for all r, i.e., to prove

Theorem 1. *Let r be a positive integer. Write n in its factoradic expansion as $n = \sum_{i=1}^{k} a_i i!$ with $1 \leq a_k \leq k$, and $0 \leq a_i \leq i$ for $i \in \{1, 2, \ldots, k-1\}$. Let*

$$S_{r,!}(n) = \sum_{i=1}^{k} a_i^r.$$

Then for $n \geq (j_r + 1)!$,

$$S_{r,!}(n) < n.$$

In our quest to prove the above theorem, we will need to study properties of the sequence j_r. In Section 2 we will prove some properties of j_r that study how the sequence grows. For example, we prove that $j_{r+1} - j_r \in \{1, 2\}$. We also prove that the number of positive integers $r \leq x$ for which $j_{r+1} - j_r = 2$ is asymptotic to $x/\log x$. Inspired by this, we say r is a *j-prime* if $j_{r+1} - j_r = 2$. In Section 3, we will prove some lemmas that are necessary for our proof of Theorem 1, in particular we have a nice upper bound for sums of powers. Finally, in Section 4, we prove Theorem 1.

2 Studying the sequence j_r

For the purposes of proving Theorem 1, we need an upper bound on j_r. The following proposition is the main result from this section we will need.

Proposition 1. *Let $\varepsilon > 0$ be a real number. Then there exists M such that, for integers $r > M$, we have that $j_r < (1 + \varepsilon)r$.*

Proof. It is enough to prove that there exists an M such that, for $r > M$,

$$\log(\lfloor(1+\varepsilon)r\rfloor!) > (r-1)\log\lfloor(1+\varepsilon)r\rfloor.$$

By expanding log as a sum we have

$$\log(\lfloor(1+\varepsilon)r\rfloor!) > \int_1^{\lfloor(1+\varepsilon)r\rfloor} \log t \, \mathrm{d}t = \lfloor(1+\varepsilon)r\rfloor \log\lfloor(1+\varepsilon)r\rfloor - \lfloor(1+\varepsilon)r\rfloor + 1.$$

Now, we want to show that there exists M such that, for $r > M$,

$$\lfloor(1+\varepsilon)r\rfloor \log\lfloor(1+\varepsilon)r\rfloor - \lfloor(1+\varepsilon)r\rfloor + 1 > (r-1)\log\lfloor(1+\varepsilon)r\rfloor,$$

which is equivalent to

$$\log\lfloor(1+\varepsilon)r\rfloor > \frac{\lfloor(1+\varepsilon)r\rfloor - 1}{\lfloor(1+\varepsilon)r\rfloor - r + 1}.$$

Since $\lfloor(1+\varepsilon)r\rfloor > (1+\varepsilon)r - 1$, then

$$\log\lfloor(1+\varepsilon)r\rfloor > \log((1+\varepsilon)r - 1),$$

and

$$\frac{\lfloor(1+\varepsilon)r\rfloor - 1}{\lfloor(1+\varepsilon)r\rfloor - r + 1} = 1 + \frac{r-2}{\lfloor(1+\varepsilon)r\rfloor - r + 1} < 1 + \frac{r-2}{\varepsilon r}.$$

If we find an M such that, for $r > M$, we have

$$\log((1+\varepsilon)r - 1) > 1 + \frac{r-2}{\varepsilon r},$$

we would finish. Note that

$$\lim_{r\to\infty} \log((1+\varepsilon)r - 1) = \infty \quad \text{and} \quad \lim_{r\to\infty}\left(1 + \frac{r-2}{\varepsilon r}\right) = 1 + \frac{1}{\varepsilon}$$

since $\varepsilon > 0$. This clearly implies the existence of the desired M. □

Remark 1. Given ε as in Proposition 1, we can take M to be

$$M = \left\lfloor \frac{e^{1+\frac{1}{\varepsilon}} + 1}{1+\varepsilon} \right\rfloor.$$

Then, for $r > M$, we have that $r > \frac{e^{1+\frac{1}{\varepsilon}}+1}{1+\varepsilon}$. Thus,

$$\log\left((1+\varepsilon)r - 1\right) > 1 + \frac{1}{\varepsilon} > 1 + \frac{1}{\varepsilon} - \frac{2}{\varepsilon r} = 1 + \frac{r-2}{\varepsilon r},$$

where the last inequality holds since $\varepsilon > 0$.

The following theorem provides an asymptotic for j_r that improves the upper bound from Proposition 1 and provides a strong lower bound. We will not need this result to prove our main theorem, but the result might be of independent interest.

Theorem 2. *For a positive integer r, there exists a real number θ_r such that*

$$j_r = r + \frac{r}{\log r} + \theta_r \left(\frac{r}{\log r}\right),$$

with $\theta_r \to 0$ as $r \to \infty$.

Proof. We will first prove the upper bound. Let $\varepsilon > 0$ be a real number, we will show for large enough r we have

$$j_r < r + \frac{(1+\varepsilon)r}{\log r}.$$

It suffices to prove that for large enough r,

$$\sum_{i=1}^{r+\left\lfloor\frac{(1+\varepsilon)r}{\log r}\right\rfloor} \log i > (r-1)\log\left(r + \left\lfloor\frac{(1+\varepsilon)r}{\log r}\right\rfloor\right). \tag{3}$$

Let $n = r + \left\lfloor \frac{(1+\varepsilon)r}{\log r} \right\rfloor$. We can bound the left side of (3) using the following explicit version of the lower bound in Stirling's formula [8], which is valid for every positive integer n:

$$\sum_{i=1}^{n} \log i > \frac{1}{2}\log(2\pi) + \left(n + \frac{1}{2}\right)\log n - n + \frac{1}{12n+1}. \tag{4}$$

Applying (4) to (3), it suffices to show

$$\log n > 1 + \frac{r - \frac{3}{2}}{\left\lfloor \frac{(1+\varepsilon)r}{\log r} \right\rfloor + \frac{3}{2}} - \frac{\log(2\pi)}{2\left(\left\lfloor \frac{(1+\varepsilon)r}{\log r} \right\rfloor + \frac{3}{2}\right)} - \frac{1}{(12n+1)\left(\left\lfloor \frac{(1+\varepsilon)r}{\log r} \right\rfloor + \frac{3}{2}\right)}.$$

Since $\frac{(1+\varepsilon)r}{\log r} \geq \left\lfloor \frac{(1+\varepsilon)r}{\log r} \right\rfloor > \frac{(1+\varepsilon)r}{\log r} - 1$, we need only prove

$$\log\left(r + \frac{(1+\varepsilon)r}{\log r} - 1\right) > 1 + \left(\frac{1}{1+\varepsilon}\right)\log r.$$

But the left side is greater than $\log r$ which is larger than $1 + \frac{1}{1+\varepsilon}\log r$ for large enough r.

For the lower bound, we will prove that for large enough r,

$$j_r > r + \frac{r}{\log r}.$$

It suffices to prove that for large enough r,

$$\sum_{i=1}^{r+\left\lfloor \frac{r}{\log r} \right\rfloor} \log i \leq (r-1)\log\left(r + \left\lfloor \frac{r}{\log r} \right\rfloor\right). \tag{5}$$

Let $n = r + \left\lfloor \frac{r}{\log r} \right\rfloor$. We can bound the left side of (5) using the following explicit version of the upper bound in Stirling's formula [8], which is valid for every positive integer n:

$$\sum_{i=1}^{n} \log i < \frac{1}{2}\log(2\pi) + \left(n + \frac{1}{2}\right)\log n - n + \frac{1}{12n}. \tag{6}$$

Applying (6) to (5), we need only show

$$\log n \leq 1 + \frac{r - \frac{3}{2}}{\left\lfloor \frac{r}{\log r} \right\rfloor + \frac{3}{2}} - \frac{\log(2\pi)}{2\left(\left\lfloor \frac{r}{\log r} \right\rfloor + \frac{3}{2}\right)} - \frac{1}{12\left(r + \left\lfloor \frac{r}{\log r} \right\rfloor\right)\left(\left\lfloor \frac{r}{\log r} \right\rfloor + \frac{3}{2}\right)}.$$

Since $\frac{r}{\log r} \geq \left\lfloor \frac{r}{\log r} \right\rfloor > \frac{r}{\log r} - 1$, letting $x = r + \frac{r}{\log r}$, we want to show that for large enough r,

$$\log x \leq 1 + \frac{r - \frac{3}{2}}{\frac{r}{\log r} + \frac{3}{2}} - \frac{\log(2\pi)}{2\left(\frac{r}{\log r} + \frac{1}{2}\right)} - \frac{1}{12(x-1)\left(\frac{r}{\log r} + \frac{1}{2}\right)}$$

or, after subtracting by $\log r$ on both sides,

$$\log\left(1+\frac{1}{\log r}\right) \leq 1 - \frac{3(\log^2 r + \log r)}{2r+3\log r} - \frac{\log(2\pi)}{2\left(\frac{r}{\log r}+\frac{1}{2}\right)} - \frac{1}{12\,(x-1)\left(\frac{r}{\log r}+\frac{1}{2}\right)}.$$

Note that the left side goes to 0 as $r \to \infty$ while the right side goes to 1 as $r \to \infty$. Therefore the inequality is true for large enough r. □

In the interest of exploring j_r further, let's analyze the relationship between j_r and j_{r+1}.

Proposition 2. *For all positive integers r, we have that $j_r + 1 \leq j_{r+1} \leq j_r + 2$.*

Proof. It is easy to see that $j_{r+1} > j_r$. So, it suffices to prove that

$$(j_r+2)! > (j_r+2)^r.$$

Assume, for the sake of contradiction, that $(j_r+2)! \leq (j_r+2)^r$. Then,

$$\frac{1}{(j_r+2)!} \geq \frac{1}{(j_r+2)^r}. \tag{7}$$

Multiplying inequalities (1) and (7),

$$\frac{j_r!}{(j_r+2)!} > \frac{j_r^{r-1}}{(j_r+2)^r}.$$

It follows that,

$$\frac{1}{j_r+1} > \frac{j_r^{r-1}}{(j_r+2)^{r-1}} = \left(\frac{j_r}{j_r+2}\right)^{r-1},$$

which implies

$$\left(1+\frac{2}{j_r}\right)^{r-1} > j_r + 1. \tag{8}$$

Since $r^{r-1} \geq r(r-1)\cdots(2) = r!$, we know $j_r \geq r$. Using this in (8),

$$\left(1+\frac{2}{r}\right)^{r-1} \geq \left(1+\frac{2}{j_r}\right)^{r-1} > j_r + 1 \geq r+1.$$

Therefore

$$\left(1+\frac{2}{r}\right)^{r-1} > r+1. \tag{9}$$

But $\left(1+\frac{2}{r}\right)^{r-1} \leq e^2 < 8$. This contradicts (9) for $r \geq 7$. We can use (2) to confirm that (9) is not satisfied for $1 \leq r \leq 6$. Thus (9) fails for $r \geq 1$. The result follows. □

Therefore, for some positive integers r we have $j_{r+1} = j_r + 1$ and for others we have $j_{r+1} = j_r + 2$. Let S be the set of positive integers such that $j_{r+1} = j_r + 2$. That is

$$S = \{r \in \mathbb{N} \mid j_{r+1} = j_r + 2\}. \tag{10}$$

The first few values in S can be easily determined from (2) to be 3, 6, 9, 12, 15, 19. This is sequence A336803. From Theorem 2 and Proposition 2 we can prove the following theorem which inspired us to name the elements of S as j-primes.

Theorem 3. *For a positive real number x, let $S(x) = \{r \leq x \mid j_{r+1} = j_r + 2\}$. Then*

$$S(x) \sim \frac{x}{\log x}.$$

Proof. From Proposition 2 we know $j_r \in \{j_{r-1} + 1, j_{r-1} + 2\}$. Therefore

$$j_r = j_1 + r - 1 + \sum_{\substack{i \in S \\ i \leq r}} 1 = r + 1 + S(r).$$

From Theorem 2 we have $j_r - r \sim \frac{r}{\log r}$. Hence $S(r) \sim \frac{r}{\log r}$. □

3 Some useful lemmas

We are interested in the sum of the first $n = j_r$ r-powers. There is an easy way to get a bound that works for all n:

$$\sum_{i=1}^{n} i^r \leq \int_1^{n+1} t^r \, \mathrm{d}t = \frac{(n+1)^{r+1} - 1}{r+1}. \tag{11}$$

For large n, (11) is great, but for n close to r, which is more relevant when $n = j_r$, we will need a tighter bound. First, we recall some known properties of the Bernoulli numbers (A027641), where we take $B_1 = \frac{1}{2}$ instead of the usual $B_1 = -\frac{1}{2}$ to simplify (14).

- For $k > 0$, $B_k < 0$ if and only if $k \equiv 0 \pmod 4$.
- We have
$$\sum_{k=0}^{\infty} \frac{B_k}{k!} = \frac{e}{e-1}. \tag{12}$$
- For $k > 0$, we have
$$B_{2k} = \frac{(-1)^{k+1} 2(2k)!}{(2\pi)^{2k}} \zeta(2k). \tag{13}$$
- $B_{2k+1} = 0$ for $k \geq 1$.

- **(Bernoulli's formula)** For positive integers n and r, we have

$$\sum_{i=1}^{n} i^r = \frac{1}{r+1}\sum_{k=0}^{r}\binom{r+1}{k}B_k n^{r+1-k}. \tag{14}$$

Lemma 1. *Let $r \geq 2$ be an integer. Then, for an integer $n \geq r+1$,*

$$\sum_{i=1}^{n} i^r \leq \frac{Cn^{r+1}}{r+1}$$

where $C = \frac{e}{e-1} + \frac{\pi^4}{45(16\pi^4-1)} = 1.583\ldots$.

Proof. For $n \geq r+1$, observe that $\binom{r+1}{k} \leq \frac{(r+1)^k}{k!} \leq \frac{n^k}{k!}$. Using that $B_{2k} < 0$ for even $k \geq 1$ and $B_k = 0$ for odd $k > 1$, by Bernoulli's formula (14) we have

$$\sum_{i=1}^{n} i^r \leq \frac{1}{r+1}\left(1 + \sum_{\substack{1\leq k\leq r \\ k \not\equiv 0 \pmod 4}} \frac{n^k}{k!}B_k n^{r+1-k}\right) \leq \frac{n^{r+1}}{r+1}\left(1 + \sum_{\substack{1\leq k\leq r \\ k \not\equiv 0 \pmod 4}} \frac{B_k}{k!}\right).$$

Now, using (12), (13), and using $\zeta(4k) \leq \zeta(4) = \frac{\pi^4}{90}$, we have

$$\begin{aligned}
1 + \sum_{\substack{1\leq k\leq r \\ k \not\equiv 0 \pmod 4}} \frac{B_k}{k!} &\leq \sum_{k=0}^{\infty}\frac{B_k}{k!} - \sum_{k=1}^{\infty}\frac{B_{4k}}{(4k)!} \\
&= \frac{e}{e-1} + 2\sum_{k=1}^{\infty}\frac{1}{(2\pi)^{4k}}\zeta(4k) \\
&\leq \frac{e}{e-1} + 2\zeta(4)\frac{1}{(2\pi)^4\left(1 - \frac{1}{(2\pi)^4}\right)} \\
&= \frac{e}{e-1} + \frac{\pi^4}{45(16\pi^4-1)}.
\end{aligned}$$

Therefore, for $n \geq r+1$,

$$\sum_{i=1}^{n} i^r \leq C\frac{n^{r+1}}{r+1},$$

where $C = \frac{e}{e-1} + \frac{\pi^4}{45(16\pi^4-1)}$, as desired. □

What follows is a simple inequality we will need to use to prove our main theorem.

Lemma 2. *Let $0 < x < r$ be a real number. Then, for all integers $r \geq 1$, we have that*

$$\left(1 - \frac{x}{r}\right)^r < e^{-x}.$$

Proof. Using the Taylor series expansion of $\log(1-x/r)$, we have that

$$r\log\left(1-\frac{x}{r}\right)=-r\left(\frac{x}{r}+\frac{1}{2}\left(\frac{x}{r}\right)^2+\frac{1}{3}\left(\frac{x}{r}\right)^3+\cdots\right)=-x-\frac{x^2}{2r}-\frac{x^3}{3r^2}-\cdots<-x.$$

□

Lemma 3. *Let $0<\varepsilon<2$. Then there exists a positive integer M such that, for all integers $r>M$, we have that*

$$\left(1-\frac{1}{j_r}\right)^{r+1}<e^{-1+\varepsilon}.$$

Proof. Let $\varepsilon_1>0$ be a real number. First, by Proposition 1, there exists M_1 positive integer such that, for $r>M_1$, we have that $j_r<(1+\varepsilon_1)r$. Then, for $r>M_1$, we have

$$\left(1-\frac{1}{j_r}\right)^{r+1}<\left(1-\frac{1}{(1+\varepsilon_1)r}\right)^{r+1}. \tag{15}$$

Observe that

$$\lim_{r\to\infty}\left(1-\frac{1}{(1+\varepsilon_1)r}\right)^{r+1}=e^{-\frac{1}{1+\varepsilon_1}}. \tag{16}$$

On the other hand, we know that $j_r>r$ because $j_1>1$ and $j_{r+1}\geq j_r+1$. It follows that

$$\left(1-\frac{1}{j_r}\right)^{r+1}>\left(1-\frac{1}{r}\right)^{r+1}, \tag{17}$$

and we have that

$$\lim_{r\to\infty}\left(1-\frac{1}{r}\right)^{r+1}=e^{-1}. \tag{18}$$

Since (15) and (16) hold for all $\varepsilon_1>0$, from (15), (16), (17), and (18) we conclude that

$$\lim_{r\to\infty}\left(1-\frac{1}{j_r}\right)^{r+1}=e^{-1}.$$

Thus, there exists M such that, for $r>M$, we have that

$$\left(1-\frac{1}{j_r}\right)^{r+1}<e^{-1+\varepsilon}$$

as desired. □

Remark 2. From Remark 1, we know that

$$j_r<\left(\frac{1}{1-\frac{\varepsilon}{2}}\right)r$$

for $r > \left(e^{\frac{2}{\varepsilon}}+1\right)\left(1-\frac{\varepsilon}{2}\right)$. Then,

$$\left(1-\frac{1}{j_r}\right)^{r+1} < \left(1-\frac{1-\frac{\varepsilon}{2}}{r}\right)^{r+1} < \left(1-\frac{1-\frac{\varepsilon}{2}}{r}\right)^{r}$$

where

$$\left(1-\frac{1-\frac{\varepsilon}{2}}{r}\right)^{r} < e^{-1+\varepsilon}$$

for all integers $r \geq 1$ (by Lemma 2 with $x = 1-\varepsilon$). Therefore, we can take M in Lemma 3 as

$$M = \left\lfloor \left(e^{\frac{2}{\varepsilon}}+1\right)\left(1-\frac{\varepsilon}{2}\right) \right\rfloor.$$

4 Proof of Theorem 1

Carlson, Goedhart, and Harris [2, Theorem 10] reduced the proof of Theorem 1 to showing the inequality[1]

$$(j_r+1)! - (j_r+1)^{r-1} + j_r! - 1 - \sum_{i=1}^{j_r-1} i^r > 0. \tag{19}$$

They proved it via computation [2, Table 1] for $2 \leq r \leq 30$. We will prove (19) works for all positive integers r in the next proposition.

Proposition 3. *Let r be a positive integer. Then*

$$(j_r+1)! - (j_r+1)^{r-1} + j_r! - 1 - \sum_{i=1}^{j_r-1} i^r > 0.$$

Proof. Let $C = \frac{e}{e-1} + \frac{\pi^4}{45(16\pi^4-1)}$ and let L_r be the left side of (19). We want to show $L_r > 0$. We know that $j_r > r$. Also, as observed by Carlson, Goedhart, and Harris [2], we have that

$$(j_r+1)^{r-1} = j_r^{r-1}\left(1+\frac{1}{j_r}\right)^{r-1} < j_r^{r-1}\left(1+\frac{1}{r}\right)^{r} < 3j_r^{r-1}.$$

From this and from Lemma 1, we have that for $r \geq 2$,

$$\begin{aligned} L_r &> j_r!(j_r+2) - 3j_r^{r-1} - 1 - \frac{C(j_r-1)^{r+1}}{r+1} \\ &= j_r!(j_r+2) - 3j_r^{r-1} - 1 - j_r^{r-1}\left(\frac{j_r^2}{r+1}\right)\left(C\left(\frac{j_r-1}{j_r}\right)^{r+1}\right) \\ &> j_r^{r-1}\left(r-1-\left(\frac{j_r^2}{r+1}\right)\left(C\left(1-\frac{1}{j_r}\right)^{r+1}\right)\right) - 1. \end{aligned}$$

[1] they incorrectly omitted the "−1" term from the inequality.

Let $0 < \varepsilon < 1 - \log(C)$ be a real number. By Lemma 3, there exists a positive integer M_1 such that, for $r > M_1$,

$$\left(1 - \frac{1}{j_r}\right)^{r+1} < e^{-1+\varepsilon},$$

which leads to

$$C\left(1 - \frac{1}{j_r}\right)^{r+1} < Ce^{-1+\varepsilon}.$$

It follows that, for $r > \max\{2, M_1\}$,

$$\begin{aligned} L_r &> j_r^{r-1}\left(r - 1 - Ce^{-1+\varepsilon}\left(\frac{j_r^2}{r+1}\right)\right) - 1 \\ &= Ce^{-1+\varepsilon} j_r^{r-1}\left(\left(\frac{1}{Ce^{-1+\varepsilon}}\right)(r-1) - \frac{j_r^2}{r+1}\right) - 1. \end{aligned}$$

Let $k > 1$ be a constant such that $k^2 < \frac{1}{Ce^{-1+\varepsilon}}$. By Proposition 1, there exists a positive integer M_2 such that, for $r > M_2$, we have that

$$j_r < k(r+1).$$

Then, for $r > \max\{2, M_1, M_2\}$,

$$L_r > Ce^{-1+\varepsilon} j_r^{r-1}\left(\left(\frac{1}{Ce^{-1+\varepsilon}} - k^2\right) r - \frac{1}{Ce^{-1+\varepsilon}} - k^2\right) - 1$$

which is greater than -1 whenever $r > \frac{1+Ck^2e^{-1+\varepsilon}}{1-Ck^2e^{-1+\varepsilon}}$. Since the expression on the left is an integer, we conclude the result for $r > \max\left\{2, M_1, M_2, \left\lfloor \frac{1+Ck^2e^{-1+\varepsilon}}{1-Ck^2e^{-1+\varepsilon}} \right\rfloor\right\}$.

From Remarks 1 and 2 we know that we can choose M_1, M_2 to be

$$\begin{aligned} M_1 &= \left\lfloor \left(e^{\frac{2}{\varepsilon}} + 1\right)\left(1 - \frac{\varepsilon}{2}\right) \right\rfloor, \\ M_2 &= \left\lfloor \frac{e^{1+\frac{1}{k-1}} + 1}{k} - 1 \right\rfloor. \end{aligned}$$

The idea is to minimize the value of $\max\left\{2, M_1, M_2, \left\lfloor \frac{1+Ck^2e^{-1+\varepsilon}}{1-Ck^2e^{-1+\varepsilon}} \right\rfloor\right\}$. Choosing $\varepsilon = 0.26$ and $k = 1.15$, we have that

$$\max\left\{2, M_1, M_2, \left\lfloor \frac{1+Ck^2e^{-1+\varepsilon}}{1-Ck^2e^{-1+\varepsilon}} \right\rfloor\right\} = \max\{2, 1907, 1857, 2167\} = 2167.$$

This proves the result for $r > 2167$. The cases $r \leq 2167$ can be easily checked with a computer. $\square$

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