A proof of a stronger Law of Sines using the Law of Cosines

Two very important theorems in geometry are the Law of Sines (LS) and the Law of Cosines (LC). Let A, B, C be the vertices of a triangle and let the lengths of the opposing sides be a, b, c, respectively. Suppose we call the angles of the triangle α, β, γ, where the angles are at the vertices A, B, C, respectively. Then LS states that

$$\frac{a}{\sin(\alpha)} = \frac{b}{\sin(\beta)} = \frac{c}{\sin(\gamma)}, \tag{1}$$

and LC states

$$a^2 = b^2 + c^2 - 2bc\cos(\alpha).$$

In [2], the author gives a simple proof of LS using the LC on three side lengths and using some nice algebraic reasoning. Below we give a different proof of a slightly stronger version of LS using LC on only one side and using a few other simple geometric facts.[1]

Theorem 1. *Let R be the circumradius of $\triangle ABC$. Then*

$$\frac{a}{\sin(\alpha)} = \frac{b}{\sin(\beta)} = \frac{c}{\sin(\gamma)} = 2R.$$

Proof. We need only show that $\frac{a}{\sin(\alpha)} = 2R$. As in Figure 1, let O be the circumcenter of $\triangle ABC$. Suppose $\alpha < 90°$. Then $\measuredangle BOC = 2\alpha$. By the Law of Cosines on $\triangle OBC$ (using that $OB = OC = R$), we have

$$a^2 = R^2 + R^2 - 2R^2\cos(2\alpha) = 2R^2(1 - (1 - 2\sin^2(\alpha))) = 4R^2(\sin(\alpha))^2.$$

Since $a, \sin(\alpha), R$ are all positive, then we can take square roots and conclude $\frac{a}{\sin(\alpha)} = 2R$. If $\alpha > 90°$, then

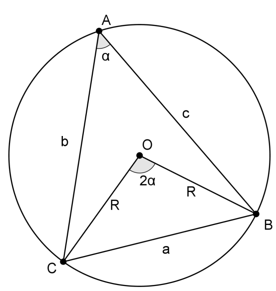

Figure 1: Triangle ABC with circumcenter O and circumradius R.

O lands outside the triangle. So the law of cosines is now used with the angle $360° - 2\alpha$ instead of 2α. But $\cos(2\alpha) = \cos(360 - 2\alpha)$, so the proof is still valid. In the case $\alpha = 90°$[2], O is the midpoint of BC, so $a = 2R$. Since $\sin(90°) = 1$, the statement $\frac{a}{\sin(\alpha)} = 2R$ follows. □

[1] We'll use that if an inscribed angle α opens an arc $\widehat{BC}$ in a circle, then the central angle opening the arc $\widehat{BC}$ is 2α. We will also use that $\cos(2\alpha) = \cos^2(\alpha) - \sin^2(\alpha) = 1 - 2\sin^2(\alpha)$.

[2] In [1], the author insists that proofs of LS should not omit the case of a right triangle.

References

[1] Alfred Brauer, *Classroom Notes: The Proof of the Law of Sines*, Amer. Math. Monthly **59** (1952), no. 5, 319. MR 1528145

[2] Patrik Nystedt, *A Proof of the Law of Sines Using the Law of Cosines*, Math. Mag. **90** (2017), no. 3, 180–181. MR 3654857