THE SMOOTHED PÓLYA–VINOGRAODOV INEQUALITY

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Abstract

Let \( \chi \) be a primitive Dirichlet character to the modulus \( q \). Let \( S_\chi(M,N) = \sum_{M<n\leq N} \chi(n) \). The Pólya-Vinogradov inequality states that \( |S_\chi(M,N)| \ll \sqrt{q} \log q \). The smoothed Pólya–Vinogradov inequality, recently introduced by Levin, Pomerance and Soundararajan, is a numerically useful version of the Pólya–Vinogradov inequality that saves a \( \log q \) factor. The smoothed Pólya–Vinogradov inequality has been used to settle a conjecture of Brizolis, namely that for every prime \( p > 3 \), there is a primitive root \( g \) and an integer \( x \in [1, p - 1] \) such that \( g^x \equiv x \mod p \). It has also been used to improve the best known numerically explicit upper bound on the least inert prime in a real quadratic field. In this paper we will prove a smoothed Pólya–Vinogradov inequality which takes into account the arithmetic properties of the modulus and we extend the inequality to imprimitive characters. We also find a lower bound for the inequality.

1. Introduction

Let \( \chi \) be a non-principal Dirichlet character to the modulus \( q \). It has been the interest of mathematicians to study the sum \( \left| \sum_{n=M+1}^{M+N} \chi(n) \right| \). Pólya and Vinogradov independently proved in 1918 that the sum is bounded above by \( O(\sqrt{q} \log q) \). Assuming the Riemann Hypothesis for L-functions (GRH), Montgomery and Vaughan [3] showed that the sum is bounded by \( O(\sqrt{q} \log \log q) \). This is best possible (up to a constant), because in 1932 Paley [5] proved that there are infinitely many quadratic characters \( \chi \) such that there exists a constant \( c > 0 \) that satisfy for some \( N \) the
inequality $\left| \sum_{n=1}^{N} \chi(n) \right| > c\sqrt{n}\log\log q$.

Recently, in [2], Levin, Pomerance and Soundararajan considered a “smoothed” version of the Pólya–Vinogradov inequality. Instead of considering the character sum over an interval, they consider the following weighted sum

$$S^*_\chi(M,N) := \sum_{M \leq n \leq M + 2N} \chi(n) \left( 1 - \left| \frac{n - M}{N} - 1 \right| \right).$$

The theorem they prove is the following:

**Theorem A.** Let $\chi$ be a primitive Dirichlet character to the modulus $q > 1$ and let $M, N$ be real numbers with $0 < N \leq q$. Then

$$\left| S^*_\chi(M,N) \right| \leq \sqrt{q} - \frac{N}{\sqrt{q}}.$$

Levin, Pomerance and Soundararajan used the inequality to prove that for every prime $p > 3$, there is a primitive root $g$ and an integer $x \in [1, p - 1]$ such that $g^x \equiv x \pmod{p}$, i.e., that the discrete logarithm base $g$ has a fixed point. The second author (see [6]) used the smoothed Pólya–Vinogradov inequality to improve an upper bound for the least inert prime in a real quadratic field. The inequality is not new, as it was used by Hua in [1] to improve a bound on the least primitive root mod $p$. However, while Hua presented his paper as an introduction of a method with numerous applications, we didn’t find other papers that used this technique. Hopefully this paper will help bring this useful method to the spotlight it deserves.

In this paper we will prove several related results. In section 2 we will prove a theorem that takes into account arithmetic information from the modulus $q$ to give a better upper bound for some ranges of $N$:

**Theorem 1.** Let $\chi$ be a primitive character to the modulus $q > 1$, let $M, N$ be real numbers with $0 < N \leq q$ and let $m$ be a divisor of $q$ such that $1 \leq m \leq \frac{q}{N}$. Then

$$\left| S^*_\chi(M,N) \right| \leq \phi(m) \sqrt{q} - \frac{N}{m\sqrt{q}}.$$

We also prove the following theorem which expands the range of $N$ and will be crucial to extend the inequality to imprimitive characters.

**Theorem 2.** Let $\chi$ be a primitive character to the modulus $q > 1$ and let $M, N$ be real numbers with $N > 0$. Then,

$$\left| \sum_{M \leq n \leq M + 2N} \chi(n) \left( 1 - \left| \frac{n - M}{N} - 1 \right| \right) \right| \leq \frac{q^{3/2}}{N} \left\{ \frac{N}{q} \right\} \left( 1 - \left\{ \frac{N}{q} \right\} \right). \quad (1)$$
In particular, $|S^*_\chi(M, N)| < \sqrt{q}$.

**Remark 1.** The theorem was stated without proof as Corollary 3 in [2]. Also note that if $0 < N < q$, then $\left\{ N \right\} = \frac{N}{q}$ and therefore Theorem A follows from Theorem 2.

With Theorem 2, we are able to extend the smoothed Pólya–Vinogradov inequality for imprimitive characters, namely we prove

**Theorem 3.** Let $\chi$ be a non-principal Dirichlet character to the modulus $q > 1$ and let $M, N$ be real numbers with $N > 0$. Then,

$$\left| \sum_{M \leq n \leq M + 2N} \chi(n) \left( 1 - \left| n - \frac{M}{N} \right| - 1 \right) \right| < \frac{4}{\sqrt{6}} \sqrt{q}.$$

One of the remarkable things involving the smoothed Pólya–Vinogradov inequality is that it is not very hard to prove and it is a tight inequality, since one can show that there exist $M$ and $N$ such that $|S^*_\chi(M, N)| > c\sqrt{q}$ for some positive constant $c$ and some character $\chi \mod q$. Indeed, in section 3 we will prove that $|S^*_\chi(M, N)| > \frac{2}{\pi} \sqrt{q}$. The proof was motivated by the proof of Theorem 9.23 in [4]. Finally, in the last section, we show a table computing $S^*_\chi(M, N)$ for many moduli.

2. Upper bound and corollaries

We begin by recreating the proof of Theorem A. We do so because the proofs of Theorem 1 and Theorem 2 branch out from this proof.

**Proof of Theorem A.** We follow the proof in [2]. Let

$$H(t) = \max\{0, 1 - |t|\}.$$

We wish to estimate $|S^*_\chi(M, N)|$.

Using the identity (see Corollary 9.8 in [4])

$$\chi(n) = \frac{1}{\tau(\chi)} \sum_{j=1}^{q} \bar{\chi}(j) e(nj/q),$$

where $e(x) := e^{2\pi ix}$ and $\tau(\chi)$ is the Gauss sum $\sum_{a=1}^{q} \chi(a) e(a/q)$, we can deduce

$$S^*_\chi(M, N) = \frac{1}{\tau(\chi)} \sum_{j=1}^{q} \bar{\chi}(j) \sum_{n \in \mathbb{Z}} e(nj/q)H \left( \frac{n - M}{N} - 1 \right).$$
The Fourier transform (see Appendix D in [4]) of \( H \) is
\[
\hat{H}(s) = \int_{-\infty}^{\infty} H(t)e(-st)dt = \frac{1 - \cos 2\pi s}{2\pi^2 s^2} \quad \text{when} \ s \neq 0, \ \hat{H}(0) = 1.
\]
Therefore \( \hat{H}(s) \) is nonnegative for \( s \) real. In general, if
\[
f(t) = e(\alpha t)H(\beta t + \gamma),
\]
with \( \beta > 0 \), then
\[
\hat{f}(s) = \frac{1}{\beta}e\left(\frac{s - \alpha}{\beta}\right)\hat{H}\left(\frac{s - \alpha}{\beta}\right).
\]
Using \( \alpha = j/q, \ \beta = 1/N \) and \( \gamma = -M/N - 1 \), then by Poisson summation (see Appendix D in [4]) we get
\[
S_{\chi}^*(M, N) = \frac{N}{\tau(\bar{\chi})} \sum_{j=1}^{q} \bar{\chi}(j) \sum_{n \in \mathbb{Z}} e\left(-(M + N)\left(n - \frac{j}{q}\right)\right) \hat{H}\left(\frac{s - \frac{j}{q}}{N}\right).
\]
Using that \( \chi(q) = 0 \), that \( \hat{H} \) is nonnegative and that \( |\tau(\bar{\chi})| = \sqrt{q} \) for primitive characters, we have
\[
|S_{\chi}^*(M, N)| \leq \frac{N}{\sqrt{q}} \sum_{j=1}^{q-1} \sum_{n \in \mathbb{Z}} H\left(\frac{n - j}{q}\right) = \frac{N}{\sqrt{q}} \sum_{k \in \mathbb{Z}/q\mathbb{Z}} \hat{H}\left(\frac{kN}{q}\right).
\]
Therefore
\[
|S_{\chi}^*(M, N)| \leq \frac{N}{\sqrt{q}} \left(\sum_{k \in \mathbb{Z}} \hat{H}\left(\frac{kN}{q}\right) - \sum_{k \in \mathbb{Z}} \hat{H}(kN)\right)
\]
\[
= \sqrt{q} \left(\sum_{k \in \mathbb{Z}} N \hat{H}\left(\frac{kN}{q}\right) - \frac{N}{q} \sum_{k \in \mathbb{Z}} \hat{H}(kN)\right)
\]
\[
\leq \sqrt{q} \left(\sum_{k \in \mathbb{Z}} N \hat{H}\left(\frac{kN}{q}\right) - \frac{N}{q} \hat{H}(0)\right).
\]
Using \( \alpha = \gamma = 0 \) and \( \beta = \frac{q}{N} \) in (2) and (3) yields that the Fourier transform of \( H\left(\frac{q}{N}\right) \) is
\[
\frac{1}{\beta}e\left(\frac{s - 0}{\beta}\right)\hat{H}\left(\frac{s - 0}{\beta}\right) = \frac{N}{q} \hat{H}\left(\frac{sN}{q}\right).
\]
Therefore, by Poisson summation, we have
\[
|S_{\chi}^*(M, N)| \leq \sqrt{q} \sum_{l \in \mathbb{Z}} H\left(\frac{ql}{N}\right) - \frac{N}{\sqrt{q}} = \sqrt{q}H(0) - \frac{N}{\sqrt{q}} = \sqrt{q} - \frac{N}{\sqrt{q}}.
\]
We used that \( q \geq N \) which implies that for \( l \neq 0 \) and \( l \in \mathbb{Z} \), \( |\frac{ql}{N}| \geq \frac{|q|}{N} \geq 1 \) which implies \( H\left(\frac{q}{N}\right) = 0. \)
Proof of Theorem 1. Following the proof of the previous theorem, we arrive at (4). From there, using that if \((n,m)>1\) then \(\chi(n)=0\), that \(\hat{H}\) is nonnegative, and that \(|\tau(\chi)|=\sqrt{q}\) for primitive characters, we have

\[
|S_\chi^\star(M,N)| \leq \frac{N}{\sqrt{q}} \sum_{j=1}^{q} \sum_{n \in \mathbb{Z}} \hat{H} \left( \left( \frac{n - \frac{j}{q}}{N} \right)^N \right) = \frac{N}{\sqrt{q}} \sum_{k \in \mathbb{Z}} \hat{H} \left( \frac{kN}{q} \right). \tag{6}
\]

Using inclusion exclusion we get

\[
|S_\chi^\star(M,N)| \leq \sqrt{q} \sum_{d \mid m} \mu(d) \sum_{k \in \mathbb{Z}} \hat{H} \left( \frac{kdN}{q} \right) = \sqrt{q} \sum_{d \mid m} \mu(d) \sum_{k \in \mathbb{Z}} \frac{dN}{q} \hat{H} \left( \frac{kdN}{q} \right).
\]

Since the Fourier transform of \(H \left( \frac{ql}{N^2} \right)\) is \(\frac{dN}{q} \hat{H} \left( \frac{sdN}{q} \right)\), by Poisson summation

\[
|S_\chi^\star(M,N)| \leq \sqrt{q} \sum_{d \mid m} \mu(d) \sum_{k \in \mathbb{Z}} H \left( \frac{ql}{N^2} \right) = \sqrt{q} \sum_{d \mid m} \mu(d) \frac{dN}{q} H(0) = \phi(m) m \sqrt{q}.
\]

We used that \(q \geq mN\) which implies that for \(l \neq 0\) and \(l \in \mathbb{Z}, \left| \frac{ql}{N^2} \right| \geq \left| \frac{qN}{m} \right| \geq 1\), and hence \(H \left( \frac{ql}{N^2} \right) = 0\). \(\square\)

Proof of Theorem 2. In the proof of Theorem A, we only used that \(N \leq q\) in the last inequality of (5). Therefore, from (5), we have

\[
|S_\chi^\star(M,N)| \leq \sqrt{q} \left( \sum_{l \in \mathbb{Z}} H \left( \frac{ql}{N} \right) - \frac{N}{q} \right).
\]

To get the desired result we need only prove

\[
\sum_{l \in \mathbb{Z}} H \left( \frac{ql}{N} \right) \leq \frac{N}{q} + \frac{q}{N} \left\{ \frac{N}{q} \right\} \left( 1 - \left\{ \frac{N}{q} \right\} \right).
\]

Note that \(H \left( \frac{ql}{N} \right) = 0\) for \(|l| > \frac{N}{q}\). Also \(H \left( \frac{ql}{N} \right) = H \left( -\frac{ql}{N} \right)\). Using these two facts together with \(H(0) = 1\), we get

\[
\sum_{l \in \mathbb{Z}} H \left( \frac{ql}{N} \right) = 1 + 2 \sum_{l \leq \frac{q}{N}} H \left( \frac{ql}{N} \right) = 1 + 2 \sum_{l \leq \frac{q}{N}} \left( 1 - \frac{ql}{N} \right).
\]

Therefore

\[
\sum_{l \in \mathbb{Z}} H \left( \frac{ql}{N} \right) = 1 + 2 \left[ \frac{N}{q} \right] - 2q \sum_{l \leq \frac{q}{N}} l = 1 + 2 \left[ \frac{N}{q} \right] - q \left( \left[ \frac{N}{q} \right] \left( \left[ \frac{N}{q} \right] + 1 \right) \right).
\]
Letting $\theta = \frac{N}{q}$ and using that $\frac{N}{q} = \left\lfloor \frac{N}{q} \right\rfloor + \theta$, we get

$$
\sum_{l \in \mathbb{Z}} H \left( \frac{ql}{N} \right) = 1 + \frac{2N}{q} - 2\theta - \frac{q}{N} \left( \frac{N^2}{q^2} + \frac{N}{q} (1 - 2\theta) - \theta (1 - \theta) \right)
$$
\[= \frac{2N}{q} + 1 - 2\theta - \frac{N}{q} - (1 - 2\theta) + \frac{q}{N} \theta (1 - \theta) = \frac{N}{q} + \frac{q}{N} \theta (1 - \theta).\]

Therefore (1) is true. Once we have (1), we can conclude that $|S_\chi^*(M, N)| < \sqrt{q}$. Indeed, if $N < q$, then

$$
S_\chi^*(M, N) \leq \frac{q^{3/2}}{N} \left\{ \frac{N}{q} \right\} \left( 1 - \left\{ \frac{N}{q} \right\} \right) = \sqrt{q} - \frac{N}{\sqrt{q}} < \sqrt{q};
$$

and if $N \geq q$, we have

$$
S_\chi^*(M, N) \leq \frac{q^{3/2}}{N} \left\{ \frac{N}{q} \right\} \left( 1 - \left\{ \frac{N}{q} \right\} \right) \leq \frac{q^{3/2}}{4N} \leq \frac{\sqrt{q}}{4} < \sqrt{q}.
$$

We finish the section with the proof of Theorem 3:

**Proof of Theorem 3.** In this proof, we follow the ideas used in [4] (page 307) to extend the Pólya–Vinogradov inequality from primitive characters to general characters.

Let $\chi$ be induced by a primitive character $\chi^*$ of modulus $d > 1$. This is possible since $\chi$ is non-principal. In the case that $\chi$ is primitive, then $\chi^* = \chi$. Letting $\chi_0$ be the principal character mod $q$, we have that $\chi = \chi^* \chi_0$. Therefore $\chi(n) = \chi^*(n)$ for $n$ an integer coprime to $q$, and $\chi(n) = 0$ otherwise.

Let $r$ be the product of primes that divide $q$ but not $d$. Then when $(n, r) > 1$, we have $\chi(n) = 0$. If $(n, r) = 1$, then $\chi(n) = \chi^*(n)$. Therefore

$$
\sum_{n=M}^{M+2N} \chi(n) \left( 1 - \left\lfloor \frac{n-M}{N} \right\rfloor - 1 \right) = \sum_{M \leq n \leq M+2N} \chi^*(n) \left( 1 - \left\lfloor \frac{n-M}{N} \right\rfloor - 1 \right)
$$
\[= \sum_{M \leq n \leq M+2N} \chi^*(n) \left( 1 - \left\lfloor \frac{n-M}{N} \right\rfloor - 1 \right) \sum_{k|n} \mu(k)
\]
\[= \sum_{k|r} \mu(k) \sum_{M \leq n \leq M+2N} \chi^*(n) \left( 1 - \left\lfloor \frac{n-M}{N} \right\rfloor - 1 \right).
\]

Now, writing $n = km$ and using that $\chi^*$ is totally multiplicative, we get

$$
\sum_{k|r} \mu(k) \chi^*(k) \sum_{\frac{M}{k} \leq m \leq \frac{M+2N}{k}} \chi^*(m) \left( 1 - \left\lfloor \frac{m-M}{N} \right\rfloor - 1 \right) = \sum_{k|r} \mu(k) \chi^*(k) S_{\chi^*} \left( \frac{M}{k}, \frac{N}{k} \right).
$$
By Theorem 2, \( |S_{\chi^*}(M/k, N/k)| < \sqrt{d} \). Hence, taking absolute value, we have
\[
|S_{\chi^*}(M, N)| < \sum_{k|\tau} \sqrt{d} = 2^{\omega(r)} \sqrt{d} \leq 2^{\omega(r)} \sqrt{\frac{q}{r}}. \tag{7}
\]
Since \( 2^{\omega(r)} \) is a multiplicative function, and for \( p \geq 5, 2 < \sqrt{p} \), we have
\[
\frac{2^{\omega(r)}}{\sqrt{r}} = \prod_{p|\tau} \frac{2}{\sqrt{p}} \leq \frac{2}{\sqrt{2}} \times \frac{2}{\sqrt{3}} = \frac{4}{\sqrt{6}}. \tag{8}
\]
Combining (7) with (8) yields the desired result.

3. Lower bound

**Theorem 4.** Let \( \chi \) be a primitive character to the modulus \( q > 1 \) and let \( M, N \) be positive integers. Then
\[
S_2(N) := \max_{1 \leq M \leq q} \left| \sum_{n=M}^{M+2N} \chi(n) \left( 1 - \frac{n-M}{N} - 1 \right) \right| \geq \frac{1}{N\sqrt{q}} \left( \frac{\sin \frac{\pi N}{q}}{\sin \frac{\pi}{q}} \right)^2. \tag{9}
\]

**Proof.** Let
\[
S_3(N) := \sum_{M=1}^{q} e\left( \frac{M}{q} \right) \sum_{n=M}^{M+2N} \chi(n) \left( 1 - \frac{n-M}{N} - 1 \right),
\]
and note that
\[
|S_3(N)| \leq \sum_{M=1}^{q} \left| \sum_{n=M}^{M+2N} \chi(n) \left( 1 - \frac{n-M}{N} - 1 \right) \right| \leq q \max_{1 \leq M \leq q} \left| \sum_{n=M}^{M+2N} \chi(n) \left( 1 - \frac{n-M}{N} - 1 \right) \right| = qS_2(N).
\]
Therefore we can focus on \( S_3(N) \).
\[
S_3(N) = \sum_{M=1}^{q} e\left( \frac{M}{q} \right) \sum_{n=M}^{M+2N} \chi(n) \left( 1 - \frac{n-M}{N} - 1 \right)
= \sum_{n=0}^{2N} \sum_{M=1}^{q} e\left( \frac{M}{q} \right) \chi(n+M) \left( 1 - \frac{n}{N} - 1 \right).
\]
Now we can do a change of variable, to go from $M$ to $L - n$:

$$S_3(N) = \sum_{n=0}^{2N} \sum_{L=1}^{q} e \left( \frac{L-n}{q} \right) \chi(L) \left( 1 - \left| \frac{n}{N} - 1 \right| \right)$$

$$= \sum_{n=0}^{2N} e \left( -\frac{n}{q} \right) \left( 1 - \left| \frac{n}{N} - 1 \right| \right) \sum_{L=1}^{q} e \left( \frac{L}{q} \right) \chi(L).$$

Therefore,

$$S_3(N) = \tau(\chi) \sum_{n=0}^{2N} e \left( -\frac{n}{q} \right) \left( 1 - \left| \frac{n}{N} - 1 \right| \right) = \tau(\chi) S_4(N).$$

Now it’s time to work on $S_4(N)$:

$$S_4(N) = \sum_{n=0}^{2N} e \left( -\frac{n}{q} \right) \left( 1 - \left| \frac{n}{N} - 1 \right| \right) = \sum_{n=0}^{N} e \left( -\frac{n}{q} \right) \frac{n}{N} + \sum_{n=N+1}^{2N} e \left( -\frac{n}{q} \right) \left( 2 - \frac{n}{N} \right).$$

By making the change of variable $m = 2N - n$, we get

$$S_4(N) = \frac{1}{N} \sum_{n=0}^{N} e \left( -\frac{n}{q} \right) n + \frac{e \left( \frac{2N}{q} \right)}{N} \sum_{m=0}^{N-1} e \left( \frac{m}{q} \right) m.$$

Using the identity

$$\sum_{n=0}^{N} nx^n = x \frac{N x^{N+1} - (N + 1) x^N + 1}{(x-1)^2} = \frac{N x^{N+1} - (N + 1) x^N + 1}{(x^{1/2} - x^{-1/2})^2},$$

with $x = e(\alpha)$ and $\alpha = -\frac{1}{q}$, we get

$$S_4(N) = \frac{1}{N} \frac{N e ((N + 1)\alpha) - (N + 1)e(\alpha) + 1}{e \left( \frac{\alpha}{2} \right) - e \left( -\frac{\alpha}{2} \right)^2} + \frac{e \left( 2\alpha \right) (N-1)e(-\alpha) - Ne((-N+1)\alpha) + 1}{N \left( e \left( -\frac{\alpha}{2} \right) - e \left( \frac{\alpha}{2} \right)^2 \right)^2}.$$

Therefore, by taking common denominator and multiplying out, we get that $S_4(N)$ equals

$$N e ((N + 1)\alpha) - (N + 1)e(\alpha) + 1 + (N-1)e(N\alpha) - Ne((N+1)\alpha) + e(2N\alpha),$$

which equals

$$\frac{e(2N\alpha) - 2e(N\alpha) + 1}{N \left( e \left( \frac{\alpha}{2} \right) - e \left( -\frac{\alpha}{2} \right) \right)^2} = \frac{e(N\alpha) \left( e \left( \frac{\alpha}{2} \right) - e \left( -\frac{\alpha}{2} \right) \right)^2}{N \left( e \left( \frac{\alpha}{2} \right) - e \left( -\frac{\alpha}{2} \right) \right)^2} = \frac{e(N\alpha) (\sin N\pi\alpha)^2}{N (\sin \pi\alpha)^2}.$$  

(10)
From earlier we know, \( qS_2(N) \geq |S_3(N)| = |\tau(\chi)||S_4(N)|. \) Using \( |\tau(\chi)| = \sqrt{q} \), that \( |e(x)| = 1 \) and (10) yields the theorem.

Now we are ready to prove our main lower bound result.

**Corollary 1.** Let \( \chi \) be a primitive character to the modulus \( q > 1 \) and let \( M, N \) be positive integers. Then

\[
\max_{M,N} \left| \sum_{n=M}^{M+2N} \chi(n) \left( 1 - \left| \frac{n - M}{N} - 1 \right| \right) \right| \geq \frac{2}{\pi^2} \sqrt{q}.
\]

**Proof.** If \( q \) is even, let \( N = \frac{q}{2} \). Therefore (9) becomes

\[
S_2(N) \geq \frac{1}{N\sqrt{q}} \left( \frac{\sin \frac{\pi N}{q}}{\sin \frac{\pi}{q}} \right)^2 = \frac{2}{q\sqrt{q}} \left( \frac{1}{\frac{\pi}{q}} \right)^2 \geq \frac{2}{\pi^2} \sqrt{q}.
\]

The last inequality comes from \( \frac{1}{\sin x} \geq \frac{1}{x} \).

If \( q \) is odd, let \( N = \frac{q-1}{2} \), then

\[
S_2(N) \geq \frac{1}{N\sqrt{q}} \left( \frac{\sin \frac{\pi N}{q}}{\sin \frac{\pi}{q}} \right)^2 = \frac{2}{(q-1)\sqrt{q}} \left( \frac{\cos \frac{\pi}{q}}{\sin \frac{\pi}{q}} \right)^2.
\]

From this and \( \sin \frac{\pi}{q} = 2 \sin \frac{\pi}{2q} \cos \frac{\pi}{2q} \), we get

\[
S_2(N) \geq \frac{2}{4(q-1)\sqrt{q}} \left( \frac{1}{\sin \frac{\pi}{2q}} \right)^2 \geq \frac{2}{\pi^2} \frac{q}{q-1} \sqrt{q} > \frac{2}{\pi^2} \sqrt{q}.
\]

**Remark 2.** If we consider \( N = \frac{q}{3} \) for \( 3 \mid q \), \( N = \frac{q-1}{3} \) for \( q \equiv 1 \pmod{3} \) and \( N = \frac{q-2}{3} \) for \( q \equiv 2 \pmod{3} \), then we can improve the constant from \( \frac{2}{\pi^2} \approx 0.202642 \) to \( \frac{9}{4\pi^2} \approx 0.227973 \). With \( N \) around \( \frac{2q}{3} \) the constant improves a bit more to \( \frac{5(5+\sqrt{5})}{16\pi^2} \approx 0.229115 \). The optimal value for \( N \) under this technique is around \( N = 0.371q \) where the constant is approximately 0.230651.

### 4. Numerics

Let \( \chi \) be a Dirichlet character mod \( q \) and let

\[
F(\chi) = \max_{M,2N \in \mathbb{Z}} \frac{|S_{\chi}^*(M,N)|}{\sqrt{q}}.
\]
Note that \( F(\chi) \) exists because \(|S_\chi(M,N)/\sqrt{q}|\) is a bounded continuous function, periodic in \( M \) and going to 0 as \( N \to \infty \) (by Theorem 2). In the previous sections we gave upper and lower bounds for \( F(\chi) \). Indeed
\[
\frac{2}{\pi^2} \leq F(\chi) < 1. \]
Now, let
\[
G(q) = \max_{\chi} F(\chi),
\]
and
\[
H(q) = \min_{\chi} F(\chi),
\]
where the max and the min range over primitive characters mod \( q \). By writing a program in Java we created the following table of values for \( G(q) \) and \( H(q) \) which show that there’s room for improvement in the upper and lower bounds, for example it seems \( \frac{2}{5} < F(q) < \frac{4}{5} \). The reason a program could be written to find \( G(q) \) and \( H(q) \) even though \( M \) and \( N \) range through all integers is that the periodicity of \( \chi \) mod \( q \) allows us to restrict ourselves to \( 0 \leq M < q \) and \( M \leq 2N < M + q \) with \( M, 2N \in \mathbb{N} \).

**Acknowledgements**

We would like to thank Carl Pomerance for his useful comments regarding this paper and his encouragement.

**References**


Table 1: A table showing the max and min of $G(q)$ and $H(q)$ for all moduli $q \leq 181$ that have primitive characters. It is worth noting that the reason the moduli divisible by 4 has such a small $G(q)$ is Theorem 1.