2. Differentiating $\sqrt[4]{x+1}$:

$$\begin{aligned}
f(x) &= \sqrt[4]{x+1} = (x+1)^{1/4} & & \\
f'(x) &= (1/4)(x+1)^{-3/4} & f(0) &= 1, \\
f''(x) &= (-3/4)(1/4)(x+1)^{-7/4} = (-3/16)(x+1)^{-7/4} & f'(0) &= \tfrac{1}{4}, \\
f'''(x) &= (-7/4)(-3/16)(x+1)^{-11/4} = (21/64)(x+1)^{-11/4} & f''(0) &= -\tfrac{3}{16}, \\
& & f'''(0) &= \tfrac{21}{64}.
\end{aligned}$$

$$\begin{aligned}
f(x) = \sqrt[4]{x+1} &= 1 + \frac{1}{4}\cdot x + \frac{(-3/16)x^2}{2!} + \frac{(21/64)x^3}{3!} + \cdots \\
&= 1 + \frac{x}{4} - \frac{3x^2}{32} + \frac{7x^3}{128} - \cdots .
\end{aligned}$$

2 points

10.

$$\begin{aligned} f(\theta) &= \cos\theta & f(\tfrac{\pi}{4}) &= \tfrac{\sqrt{2}}{2}, \\ f'(\theta) &= -\sin\theta & f'(\tfrac{\pi}{4}) &= -\tfrac{\sqrt{2}}{2}, \\ f''(\theta) &= -\cos\theta & f''(\tfrac{\pi}{4}) &= -\tfrac{\sqrt{2}}{2}, \\ f'''(\theta) &= \sin\theta & f'''(\tfrac{\pi}{4}) &= \tfrac{\sqrt{2}}{2}. \end{aligned}$$

$$\begin{aligned} \cos\theta &= \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\left(\theta - \frac{\pi}{4}\right) - \frac{\sqrt{2}}{2}\frac{(\theta-\frac{\pi}{4})^2}{2!} + \frac{\sqrt{2}}{2}\frac{(\theta-\frac{\pi}{4})^3}{3!} - \cdots \\ &= \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\left(\theta - \frac{\pi}{4}\right) - \frac{\sqrt{2}}{4}\left(\theta - \frac{\pi}{4}\right)^2 + \frac{\sqrt{2}}{12}\left(\theta - \frac{\pi}{4}\right)^3 - \cdots \end{aligned}$$

11. Differentiating gives

$$\begin{aligned} f(t) &= \cos t & f(\tfrac{\pi}{6}) &= \tfrac{\sqrt{3}}{2}, \\ f'(t) &= -\sin t & f'(\tfrac{\pi}{6}) &= -\tfrac{1}{2}, \\ f''(t) &= -\cos t & f''(\tfrac{\pi}{6}) &= -\tfrac{\sqrt{3}}{2}, \\ f'''(t) &= \sin t & f'''(\tfrac{\pi}{6}) &= \tfrac{1}{2}. \end{aligned}$$

$$\begin{aligned} \cos t &= \frac{\sqrt{3}}{2} - \frac{1}{2}\left(t - \frac{\pi}{6}\right) - \frac{\sqrt{3}}{2}\frac{(t-\frac{\pi}{6})^2}{2!} + \frac{1}{2}\frac{(t-\frac{\pi}{6})^3}{3!} - \cdots \\ &= \frac{\sqrt{3}}{2} - \frac{1}{2}\left(t - \frac{\pi}{6}\right) - \frac{\sqrt{3}}{4}\left(t - \frac{\pi}{6}\right)^2 + \frac{1}{12}\left(t - \frac{\pi}{6}\right)^3 - \cdots \end{aligned}$$

12.

$$\begin{aligned} f(\theta) &= \sin\theta & f(-\tfrac{\pi}{4}) &= -\tfrac{\sqrt{2}}{2}, \\ f'(\theta) &= \cos\theta & f'(-\tfrac{\pi}{4}) &= \tfrac{\sqrt{2}}{2}, \\ f''(\theta) &= -\sin\theta & f''(-\tfrac{\pi}{4}) &= \tfrac{\sqrt{2}}{2}, \\ f'''(\theta) &= -\cos\theta & f'''(-\tfrac{\pi}{4}) &= -\tfrac{\sqrt{2}}{2}. \end{aligned}$$

$$\begin{aligned} \sin\theta &= -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(\theta + \frac{\pi}{4}\right) + \frac{\sqrt{2}}{2}\frac{(\theta+\frac{\pi}{4})^2}{2!} - \frac{\sqrt{2}}{2}\frac{(\theta+\frac{\pi}{4})^3}{3!} + \cdots \\ &= -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(\theta + \frac{\pi}{4}\right) + \frac{\sqrt{2}}{4}\left(\theta + \frac{\pi}{4}\right)^2 - \frac{\sqrt{2}}{12}\left(\theta + \frac{\pi}{4}\right)^3 + \cdots. \end{aligned}$$

13.

$$\begin{aligned} f(x) &= \tan x & f(\tfrac{\pi}{4}) &= 1, \\ f'(x) &= \tfrac{1}{\cos^2 x} & f'(\tfrac{\pi}{4}) &= 2, \\ f''(x) &= \tfrac{-2(-\sin x)}{\cos^3 x} = \tfrac{2\sin x}{\cos^3 x} & f''(\tfrac{\pi}{4}) &= 4, \\ f'''(x) &= \tfrac{-6\sin x(-\sin x)}{\cos^4 x} + \tfrac{2}{\cos^2 x} & f'''(\tfrac{\pi}{4}) &= 16. \end{aligned}$$

$$\begin{aligned} \tan x &= 1 + 2\left(x - \frac{\pi}{4}\right) + 4\frac{(x-\frac{\pi}{4})^2}{2!} + 16\frac{(x-\frac{\pi}{4})^3}{3!} + \cdots \\ &= 1 + 2\left(x - \frac{\pi}{4}\right) + 2\left(x - \frac{\pi}{4}\right)^2 + \frac{8}{3}\left(x - \frac{\pi}{4}\right)^3 + \cdots \end{aligned}$$

14.

$$\begin{aligned} f(x) &= \tfrac{1}{x} & f(1) &= 1 \\ f'(x) &= -\tfrac{1}{x^2} & f'(1) &= -1 \\ f''(x) &= \tfrac{2}{x^3} & f''(1) &= 2 \\ f'''(x) &= -\tfrac{6}{x^4} & f'''(1) &= -6 \end{aligned}$$

$$\begin{aligned}\frac{1}{x} &= 1-(x-1)+\frac{2(x-1)^2}{2!}-\frac{6(x-1)^3}{3!}+\cdots \\ &= 1-(x-1)+(x-1)^2-(x-1)^3+\cdots.\end{aligned}$$

15. Again using the derivatives found in Problem 14, we have

$$f(2)=\frac{1}{2},\qquad f'(2)=-\frac{1}{4},\qquad f''(2)=\frac{1}{4},\qquad f'''(2)=-\frac{3}{8}.$$

$$\begin{aligned}\frac{1}{x} &= \frac{1}{2}-\frac{x-2}{4}+\frac{(x-2)^2}{4\cdot 2!}-\frac{3(x-2)^3}{8\cdot 3!}+\cdots \\ &= \frac{1}{2}-\frac{(x-2)}{4}+\frac{(x-2)^2}{8}-\frac{(x-2)^3}{16}+\cdots\end{aligned}$$

16. Using the derivatives from Problem 14, we have

$$f(-1)=-1,\quad f'(-1)=-1,\quad f''(-1)=-2,\quad f'''(-1)=-6.$$

Hence,

$$\begin{aligned}\frac{1}{x} &= -1-(x+1)-\frac{2(x+1)^2}{2!}-\frac{6(x+1)^3}{3!}-\cdots \\ &= -1-(x+1)-(x+1)^2-(x+1)^3-\cdots\end{aligned}$$

17. The general term can be written as x^n for $n \geq 0$.

18. The general term can be written as $(-1)^n x^n$ for $n \geq 0$.

19. The general term can be written as $-x^n/n$ for $n \geq 1$.

20. The general term can be written as $(-1)^{n-1}x^n/n$ for $n \geq 1$.

21. The general term can be written as $(-1)^k x^{2k+1}/(2k+1)!$ for $k \geq 0$.

22. The general term can be written as $(-1)^k x^{2k+1}/(2k+1)$ for $k \geq 0$.

23. The general term can be written as $x^{2k}/k!$ for $k \geq 0$.

2 points

30. The second coefficient of the Taylor expansion is

$$\frac{g''(0)}{2} = 1, \quad \text{so} \quad g''(0) = 2.$$

Similarly, the third coefficient is

$$\frac{g'''(0)}{3!} = 0 \quad \text{so} \quad g'''(0) = 0.$$

Finally, the tenth coefficient is

$$\frac{g^{(10)}(0)}{10!} = \frac{1}{5!} \quad \text{so} \quad g^{(10)}(0) = \frac{10!}{5!}.$$

31. Let C_n be the coefficient of the n^{th} term in the series. Note that

Using the fact that $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, $i^5 = i, \cdots$, we can rewrite the series as

$$\cos\theta + i\sin\theta = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \frac{(i\theta)^6}{6!} + \cdots$$

Amazingly enough, this series is the Taylor series for e^x with $i\theta$ substituted for x. Therefore, we have shown that

$$\cos\theta + i\sin\theta = e^{i\theta}.$$

34. This is the series for e^x with x replaced by 2, so the series converges to e^2.

35. This is the series for $\sin x$ with x replaced by 1, so the series converges to $\sin 1$.

36. This is the series for $1/(1-x)$ with x replaced by $1/4$, so the series converges to $1/(1-(1/4)) = 4/3$.

37. This is the series for $\cos x$ with x replaced by 10, so the series converges to $\cos 10$.

38. This is the series for $\ln(1+x)$ with x replaced by $1/2$, so the series converges to $\ln(3/2)$.

39. The Taylor series for $f(x) = 1/(1+x)$ is

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots.$$

Substituting $x = 0.1$ gives

$$1 - 0.1 + (0.1)^2 - (0.1)^3 + \cdots = \frac{1}{1+0.1} = \frac{1}{1.1}.$$

Alternatively, this is a geometric series with $a = 1$, $x = -0.1$.

40. This is the series for e^x with $x = 3$ substituted. Thus

$$1 + 3 + \frac{9}{2!} + \frac{27}{3!} + \frac{81}{4!} + \cdots = 1 + 3 + \frac{3^2}{2!} + \frac{3^3}{3!} + \frac{3^4}{4!} + \cdots = e^3.$$

41. This is the series for $\cos x$ with $x = 1$ substituted. Thus

$$1 - \frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!} + \cdots = \cos 1.$$

42. This is the series for e^x with -0.1 substituted for x, so

$$1 - 0.1 + \frac{0.01}{2!} - \frac{0.001}{3!} + \cdots = e^{-0.1}.$$

43. Since $1 + x + x^2 + x^3 + \cdots = \dfrac{1}{1-x}$, a geometric series, we solve $\dfrac{1}{1-x} = 5$ giving $\dfrac{1}{5} = 1 - x$, so $x = \dfrac{4}{5}$.

44. Since $x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots = \ln(1+x)$, we solve $\ln(1+x) = 0.2$, giving $1 + x = e^{0.2}$, so $x = e^{0.2} - 1$.

Solutions for Section 10.3

Exercises

1. Substitute $y = -x$ into $e^y = 1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \cdots$. We get

$$\begin{aligned} e^{-x} &= 1 + (-x) + \frac{(-x)^2}{2!} + \frac{(-x)^3}{3!} + \cdots \\ &= 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots. \end{aligned}$$

10.3

2. We'll use

$$\sqrt{1+y} = (1+y)^{\frac{1}{2}} = 1 + \left(\frac{1}{2}\right) y + \left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)\frac{y^2}{2!} + \left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)\frac{y^3}{3!} + \cdots$$
$$= 1 + \frac{y}{2} - \frac{y^2}{8} + \frac{y^3}{16} - \cdots.$$

Substitute $y = -2x$.

$$\sqrt{1-2x} = 1 + \frac{(-2x)}{2} - \frac{(-2x)^2}{8} + \frac{(-2x)^3}{16} - \cdots$$
$$= 1 - x - \frac{x^2}{2} - \frac{x^3}{2} - \cdots$$

8.

$$\frac{z}{e^{z^2}} = ze^{-z^2} = z\left(1 + (-z^2) + \frac{(-z^2)^2}{2!} + \frac{(-z^2)^3}{3!} + \cdots\right)$$
$$= z - z^3 + \frac{z^5}{2!} - \frac{z^7}{3!} + \cdots$$

9.

$$\phi^3\cos(\phi^2) = \phi^3\left(1 - \frac{(\phi^2)^2}{2!} + \frac{(\phi^2)^4}{4!} - \frac{(\phi^2)^6}{6!} + \cdots\right)$$
$$= \phi^3 - \frac{\phi^7}{2!} + \frac{\phi^{11}}{4!} - \frac{\phi^{15}}{6!} + \cdots$$

10. Substituting the series for $\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots$ into

$$\sqrt{1+y} = 1 + \frac{1}{2}y - \frac{1}{8}y^2 + \frac{1}{16}y^3 - \cdots$$

gives

$$\sqrt{1+\sin\theta} = 1 + \frac{1}{2}\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right) - \frac{1}{8}\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)^2$$
$$+ \frac{1}{16}\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)^3 - \cdots$$
$$= 1 + \frac{1}{2}\theta - \frac{\theta^2}{8} + \left(\frac{\theta^3}{16} - \frac{\theta^3}{2\cdot 3!}\right) + \cdots$$
$$= 1 + \frac{1}{2}\theta - \frac{1}{8}\theta^2 - \frac{1}{48}\theta^3 + \cdots$$

11.

$$\sqrt{(1+t)}\sin t = \left(1 + \frac{t}{2} - \frac{t^2}{8} + \frac{t^3}{16} - \cdots\right)\left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \cdots\right)$$

Multiplying and collecting terms yields

$$\sqrt{(1+t)}\sin t = t + \frac{t^2}{2} - \left(\frac{t^3}{3!} + \frac{t^3}{8}\right) + \left(\frac{t^4}{16} - \frac{t^4}{12}\right) + \cdots$$
$$= t + \frac{1}{2}t^2 - \frac{7}{24}t^3 - \frac{1}{48}t^4 + \cdots.$$

2 points

12.

$$e^t\cos t = \left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots\right)\left(1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \cdots\right)$$

Multiplying out and collecting terms gives

$$e^t\cos t = 1 + t + \left(\frac{t^2}{2!} - \frac{t^2}{2!}\right) + \left(\frac{t^3}{3!} - \frac{t^3}{2!}\right) + \left(\frac{t^4}{4!} + \frac{t^4}{4!} - \frac{t^4}{(2!)^2}\right) + \cdots$$
$$= 1 + t - \frac{t^3}{3} - \frac{t^4}{6} + \cdots.$$

13. Multiplying out gives $(1+x)^3 = 1 + 3x + 3x^2 + x^3$. Since this polynomial equals the original function for all x, it must be the Taylor series. The general term is $0 \cdot x^n$ for $n \geq 4$.

(b) For x near 0, we can approximate $e^x - e^{-x}$ by its third degree Taylor polynomial, $P_3(x)$:

$$e^x - e^{-x} \approx P_3(x) = 2x + \frac{x^3}{3}.$$

2 points The function $P_3(x)$ is a cubic polynomial whose graph is symmetric about the origin.

24. Notice that $\sum px^{p-1}$, is the derivative, term-by-term, of a geometric series:

$$\sum_{p=1}^{\infty} px^{p-1} = 1 \cdot x^0 + 2 \cdot x^1 + 3 \cdot x^2 + \cdots = \frac{d}{dx}\underbrace{(x + x^2 + x^3 + \cdots)}_{\text{Geometric series}}.$$

For $|x| < 1$, the sum of the geometric series with first term x and common ratio x is

$$x + x^2 + x^3 + \cdots = \frac{x}{1-x}.$$

Differentiating gives

$$\sum_{p=1}^{\infty} px^{p-1} = \frac{d}{dx}\left(\frac{x}{1-x}\right) = \frac{1(1-x) - x(-1)}{(1-x)^2} = \frac{1}{(1-x)^2}.$$

25. From the series for $\ln(1+y)$,

$$\ln(1+y) = y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \cdots,$$

we get

$$\ln(1+y^2) = y^2 - \frac{y^4}{2} + \frac{y^6}{3} - \frac{y^8}{4} + \cdots$$

The Taylor series for $\sin y$ is

$$\sin y = y - \frac{y^3}{3!} + \frac{y^5}{5!} - \frac{y^7}{7!} + \cdots$$

So

$$\sin y^2 = y^2 - \frac{y^6}{3!} + \frac{y^{10}}{5!} - \frac{y^{14}}{7!} + \cdots$$

The Taylor series for $\cos y$ is

$$\cos y = 1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \frac{y^6}{6!} + \cdots$$

So

$$1 - \cos y = \frac{y^2}{2!} - \frac{y^4}{4!} + \frac{y^6}{6!} + \cdots$$

Near $y = 0$, we can drop terms beyond the fourth degree in each expression:

$$\ln(1+y^2) \approx y^2 - \frac{y^4}{2}$$
$$\sin y^2 \approx y^2$$
$$1 - \cos y \approx \frac{y^2}{2!} - \frac{y^4}{4!}.$$

(Note: These functions are all even, so what holds for negative y will hold for positive y.) Clearly $1 - \cos y$ is smallest, because the y^2 term has a factor of $\frac{1}{2}$. Thus, for small y,

$$\frac{y^2}{2!} - \frac{y^4}{4!} < y^2 - \frac{y^4}{2} < y^2$$

so

$$1 - \cos y < \ln(1+y^2) < \sin(y^2).$$

2 points

26. The Taylor series about 0 for $y = \dfrac{1}{1-x^2}$ is

$$y = 1 + x^2 + x^4 + x^6 + \cdots.$$

The series for $y = (1+x)^{1/4}$ is, using the binomial expansion,

$$y = 1 + \frac{1}{4}x + \frac{1}{4}\left(-\frac{3}{4}\right)\frac{x^2}{2!} + \frac{1}{4}\left(-\frac{3}{4}\right)\left(-\frac{7}{4}\right)\frac{x^3}{3!} + \cdots.$$

The series for $y = \sqrt{1+\frac{x}{2}} = (1+\frac{x}{2})^{1/2}$ is, again using the binomial expansion,

$$y = 1 + \frac{1}{2}\cdot\frac{x}{2} + \frac{1}{2}\left(-\frac{1}{2}\right)\cdot\frac{x^2}{8} + \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\cdot\frac{x^3}{48} + \cdots.$$

Similarly for $y = \dfrac{1}{\sqrt{1-x}} = (1-x)^{-(1/2)}$,

$$y = 1 + \left(-\frac{1}{2}\right)(-x) + \left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\cdot\frac{x^2}{2!} + \left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\cdot\frac{-x^3}{3!} + \cdots.$$

Near 0, let's truncate these series after their x^2 terms:

$$\frac{1}{1-x^2} \approx 1 + x^2,$$

$$(1+x)^{1/4} \approx 1 + \frac{1}{4}x - \frac{3}{32}x^2,$$

$$\sqrt{1+\frac{x}{2}} \approx 1 + \frac{1}{4}x - \frac{1}{32}x^2,$$

$$\frac{1}{\sqrt{1-x}} \approx 1 + \frac{1}{2}x + \frac{3}{8}x^2.$$

Thus $\frac{1}{1-x^2}$ looks like a parabola opening upward near the origin, with y-axis as the axis of symmetry, so (a) = I.

Now $\frac{1}{\sqrt{1-x}}$ has the largest positive slope ($\frac{1}{2}$), and is concave up (because the coefficient of x^2 is positive). So (d) = II.

The last two both have positive slope ($\frac{1}{4}$) and are concave down. Since $(1+x)^{\frac{1}{4}}$ has the smallest second derivative (i.e., the most negative coefficient of x^2), (b) = IV and therefore (c) = III.

27. Since it does not depend ...