relatively

powers of

**2.** The error bound in approximating $\sin(0.2)$ using the Taylor polynomial of degree 3 for $f(x) = \sin x$ about $x = 0$ is:

$$|E_3| = |f(0.2) - P_3(0.2)| \leq \frac{M \cdot |0.2 - 0|^4}{4!} = \frac{M(0.2)^4}{24},$$

where $|f^{(4)}(x)| \leq M$ for $0 \leq x \leq 0.2$. Now, $f^{(4)}(x) = \sin x$. By looking at the graph of $\sin x$, we see that $|f^{(4)}(x)|$ is maximized for $x$ between 0 and 0.2 when $x = 0.2$. Thus,

$$|f^{(4)}| \leq \sin(0.2),$$

so

$$|E_3| \leq \frac{\sin(0.2) \cdot (0.2)^4}{24} = 0.0000132.$$

The Taylor polynomial of degree 3 is

$$P_3(x) = x - \frac{1}{3!}x^3.$$

The approximation is $P_3(0.1)$, so the actual error is

$$E_3 = \sin(0.2) - P_3(0.2) = 0.19866933 - 0.19866667 = 0.00000266,$$

which is much less than the bound.

2 points which is much less than the bound.

**8.** The error bound in approximating $0.5^{1/3}$ using the Taylor polynomial of degree 3 for $f(x) = (1-x)^{1/3}$ about $x = 0$ is:

$$|E_3| = |f(0.5) - P_3(0.5)| \leq \frac{M \cdot |0.5 - 0|^4}{4!} = \frac{M(0.5)^4}{24},$$

out where $|f^{(4)}(x)| \leq M$ for $0 \leq x \leq 0.5$. Now,

$$f^{(4)}(x) = -\frac{80}{81}(1-x)^{-11/3}.$$

By looking at the graph of $(1-x)^{-11/3}$, we see that $|f^{(4)}(x)|$ is maximized for $x$ between 0 and 0.5 when $x = 0.5$. Thus,

$$|f^{(4)}| \leq \frac{80}{81}\left(\frac{1}{2}\right)^{-11/3} = \frac{80}{81} \cdot 2^{11/3},$$

so

$$|E_3| \leq \frac{80 \cdot 2^{11/3} \cdot (0.5)^4}{81 \cdot 24} = 0.033.$$

2. The Taylor polynomial of degree 3 is

$$\begin{aligned} P_3(x) &= 1 + \frac{1}{3}(-x) + \frac{1}{3}\frac{-2}{3}\frac{(-x)^2}{2!} + \frac{1}{3}\frac{-2}{3}\frac{-5}{3}\frac{(-x)^3}{3!} \\ &= 1 - \frac{1}{3}x - \frac{1}{9}x^2 - \frac{5}{81}x^3. \end{aligned}$$

The approximation is $P_3(0.5)$, so the actual error is

$$E_3 = (0.5)^{1/3} - P_3(0.5) = 0.79370 - 0.79784 = -0.00414,$$

which is much less than the bound.

2 points

$$|E_4| \le \frac{}{120} \le 0.0084.$$

12. (a) The Taylor polynomial of degree 0 about $t = 0$ for $f(t) = e^t$ is simply $P_0(x) = 1$. Since $e^t \geq 1$ on $[0, 0.5$ approximation is an underestimate.

(b) Using the zero degree error bound, if $|f'(t)| \leq M$ for $0 \leq t \leq 0.5$, then

$$|E_0| \leq M \cdot |t| \leq M(0.5).$$

Since $|f'(t)| = |e^t| = e^t$ is increasing on $[0, 0.5]$,

$$|f'(t)| \leq e^{0.5} < \sqrt{4} = 2.$$

Therefore

$$|E_0| \leq (2)(0.5) = 1.$$

(Note: By looking at a graph of $f(t)$ and its $0^{\text{th}}$ degree approximation, it is easy to see that the greatest error oc when $t = 0.5$, and the error is $e^{0.5} - 1 \approx 0.65 < 1$. So our error bound works.)

13. (a) The second degree Taylor

2 points



**14.** (a) (i) The vertical distance between the graph of $y = \cos x$ and $y = P_{10}(x)$ at $x = 6$ is no more than 4, so

$$|\text{Error in } P_{10}(6)| \leq 4.$$

Since at $x = 6$ the $\cos x$ and $P_{20}(x)$ graphs are indistinguishable in this figure, the error must be less than the smallest division we can see, which is about 0.2 so,

$$|\text{Error in } P_{20}(6)| \leq 0.2.$$

(ii) The maximum error occurs at the ends of the interval, that is, at $x = -9, x = 9$. At $x = 9$, the graphs of $y = \cos x$ and $y = P_{20}(x)$ are no more than 1 apart, so

$$\left|\begin{array}{l}\text{Maximum error in } P_{20}(x)\\ \text{for } -9 \leq x \leq 9\end{array}\right| \leq 1.$$

(b) We are looking for the largest $x$-interval on which the graphs of $y = \cos x$ and $y = P_{10}(x)$ are indistinguishable. This is hard to estimate accurately from the figure, though $-4 \leq x \leq 4$ certainly satisfies this condition.

**15.** The maximum possible error for the $n^{\text{th}}$ degree Taylor polynomial about $x = 0$ approximating $\cos x$ is $|E_n| \leq \frac{M\cdot|x-0|^{n+1}}{(n+1)!}$, where $|\cos^{(n+1)} x| \leq M$ for $0 \leq x \leq 1$. Now the derivatives of $\cos x$ are simply $\cos x, \sin x, -\cos x$, and $-\sin x$. The largest magnitude these ever take is 1, so $|\cos^{(n+1)}(x)| \leq 1$, and thus $|E_n| \leq \frac{|x|^{n+1}}{(n+1)!} \leq \frac{1}{(n+1)!}$. The same argument works for $\sin x$.

**16.** By the results of Problem 15, if we approximate $\cos 1$ using the $n^{\text{th}}$ degree polynomial, the error is at most $\frac{1}{(n+1)!}$. For the answer to be correct to four decimal places, the error must be less than 0.00005. Thus, the first $n$ such that $\frac{1}{(n+1)!} < 0.00005$ will work. In particular, when $n = 7$, $\frac{1}{8!} = \frac{1}{40370} < 0.00005$, so the $7^{\text{th}}$ degree Taylor polynomial will give the desired result. For six decimal places, we need $\frac{1}{(n+1)!} < 0.0000005$. Since $n = 9$ works, the $9^{\text{th}}$ degree Taylor polynomial is sufficient.

**17.** (a)