4. We use the substitution $w = 3x$, $dw = 3\,dx$.

$$\int e^{3x}\,dx = \frac{1}{3}\int e^{w}\,dw = \frac{1}{3}e^{w} + C = \frac{1}{3}e^{3x} + C.$$

Check: $\frac{d}{dx}\left(\frac{1}{3}e^{3x} + C\right) = \frac{1}{3}e^{3x}(3) = e^{3x}$.

12. We use the substitution $w = y^2 + 5$, $dw = 2y\,dy$.

$$\int y(y^2+5)^8\,dy = \frac{1}{2}\int (y^2+5)^8(2y\,dy)$$

$$= \frac{1}{2}\int w^8\,dw = \frac{1}{2}\frac{w^9}{9} + C$$

$$= \frac{1}{18}(y^2+5)^9 + C.$$

Check: $\frac{d}{dy}\left(\frac{1}{18}(y^2+5)^9 + C\right) = \frac{1}{18}[9(y^2+5)^8(2y)] = y(y^2+5)^8.$

13. We use the substitution $w = t^3 - 3$, $dw = 3t^2\,dt$.

32. We use the substitution $w = \sqrt{x}$, $dw = \frac{1}{2\sqrt{x}}\,dx$.

$$\int \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx = \int \cos w(2\,dw) = 2\sin w + C = 2\sin\sqrt{x} + C.$$

Check: $\frac{d}{dx}(2\sin\sqrt{x} + C) = 2\cos\sqrt{x}\left(\frac{1}{2\sqrt{x}}\right) = \frac{\cos\sqrt{x}}{\sqrt{x}}.$

33. We use the substitution $w = \sqrt{y}$, $dw = \dfrac{1}{2\sqrt{y}}\,dy$.

$$\int \frac{e^{\sqrt{y}}}{\sqrt{y}}\,dy = 2\int e^w\,dw = 2e^w + C = 2e^{\sqrt{y}} + C.$$

Check: $\frac{d}{dy}(2e^{\sqrt{y}} + C) = 2e^{\sqrt{y}} \cdot \frac{1}{2\sqrt{y}} = \frac{e^{\sqrt{y}}}{\sqrt{y}}.$

34. We use the substitution $w = x + e^x$, $dw = (1 + e^x)\,dx$.

$$\int \frac{1 + e^x}{\sqrt{x + e^x}}\,dx = \int \frac{dw}{\sqrt{w}} = 2\sqrt{w} + C = 2\sqrt{x + e^x} + C.$$

Check: $\frac{d}{dx}(2\sqrt{x + e^x} + C) = 2 \cdot \frac{1}{2}(x + e^x)^{-\frac{1}{2}} \cdot (1 + e^x) = \frac{1 + e^x}{\sqrt{x + e^x}}.$

35. We use the substitution $w = 2 + e^x$, $dw = e^x\,dx$.

$$\int \frac{e^x}{2 + e^x}\,dx = \int \frac{dw}{w} = \ln|w| + C = \ln(2 + e^x) + C.$$

(We can drop the absolute value signs since $2 + e^x \geq 0$ for all x.)

Check: $\frac{d}{dx}[\ln(2 + e^x) + C] = \frac{1}{2 + e^x} \cdot e^x = \frac{e^x}{2 + e^x}.$

36. We use the substitution $w = x^2 + 2x + 19$, $dw = 2(x + 1)dx$.

$$\int \frac{(x + 1)dx}{x^2 + 2x + 19} = \frac{1}{2}\int \frac{dw}{w} = \frac{1}{2}\ln|w| + C = \frac{1}{2}\ln(x^2 + 2x + 19) + C.$$

(We can drop the absolute value signs, since $x^2 + 2x + 19 = (x + 1)^2 + 18 > 0$ for all x.)

Check: $\frac{1}{dx}[\frac{1}{2}\ln(x^2 + 2x + 19)] = \frac{1}{2}\frac{1}{x^2 + 2x + 19}(2x + 2) = \frac{x + 1}{x^2 + 2x + 19}.$

$\Big|_0 \qquad e$

58. $\displaystyle\int_1^2 2xe^{x^2}\,dx = e^{x^2}\Big|_1^2 = e^{2^2} - e^{1^2} = e^4 - e = \boxed{e(e^3 - 1)}$

59. We substitu

80. If we let $u = \pi - x$ in the first integral, we get $dx = -du$ and $x = \pi - u$. Also, the limits $x = 0$ and $x = \pi$ become $u = \pi$ and $u = 0$. Thus

$$\int_0^{\pi} (\pi - x)\cos x \, dx = -\int_{\pi}^{0} u\cos(\pi - u)\, du = \int_0^{\pi} u\cos(\pi - u)\, du.$$

If we rename the variable u as x in the last integral, we have the equality we want.

81. As x goes from $\sqrt{a}$ to $\sqrt{b}$ the values of $u = x^2$ increase from a to b. Since $x = \sqrt{u}$ we have $dx = du/(2\sqrt{u})$. Hence

94. Since e^{t+1} is larger than e^t, we have

$$\text{Area} = \int_0^2 (e^{t+1} - e^t)\,dt = (e^{t+1} - e^t)\Big|_0^2 = e^3 - e^2 - e + 1.$$

(The integral $\int e^{t+1}\,dt = e^{t+1} + C$ can be done by substitution or guess and check.)

95. The curves $y = e^x$ and $y = 3$ cross where

102. (a) $\int 4x(x^2+1)\,dx = \int (4x^3+4x)\,dx = x^4 + 2x^2 + C.$

(b) If $w = x^2+1$, then $dw = 2x\,dx$.

$$\int 4x(x^2+1)\,dx = \int 2w\,dw = w^2 + C = (x^2+1)^2 + C.$$

(c) The expressions from parts (a) and (b) look different, but they are both correct. Note that $(x^2+1)^2 + C = x^4 + 2x^2 + 1 + C$. In other words, the expressions from parts (a) and (b) differ only by a constant, so they are both correct antiderivatives.

103. (a) We first try the substitution $w = \sin\theta$, $dw = \cos\theta\,d\theta$. Then

108. (a) In 2010, we have $P = 6.1e^{0.012 \cdot 10} = 6.9$ billion people.
In 2020, we have $P = 6.1e^{0.012 \cdot 20} = 7.8$ billion people.

(b) We have

$$\text{Average population} = \frac{1}{10-0}\int_0^{10} 6.1e^{0.012t}\,dt = \frac{1}{10}\cdot\frac{6.1}{0.012}e^{0.012t}\Big|_0^{10}$$

$$= \frac{1}{10}\left(\frac{6.1}{0.012}(e^{0.12}-e^{0})\right) = 6.5.$$

The average population of the world between 2000 and 2010 is predicted to be 6.5 billion people.

109. (a) At time $t = 0$, the rate of oil leakage $= r(0) = 50$ thousand liters/minute.

4. Let $u = t^2$, $v' = \sin t$ implying $v = -\cos t$ and $u' = 2t$. Integrating by parts, we get:

$$\int t^2 \sin t \, dt = -t^2 \cos t - \int 2t(-\cos t) \, dt.$$

Again, applying integration by parts with $u = t$, $v' = \cos t$, we have:

$$\int t \cos t \, dt = t \sin t + \cos t + C.$$

Thus

$$\int t^2 \sin t \, dt = -t^2 \cos t + 2t \sin t + 2 \cos t + C.$$

12. Let $u = \theta^2$ and $v' = \cos 3\theta$, so $u' = 2\theta$ and $v = \frac{1}{3}\sin 3\theta$.

Then $\int \theta^2 \cos 3\theta \, d\theta = \frac{1}{3}\theta^2 \sin 3\theta - \frac{2}{3}\int \theta \sin 3\theta \, d\theta$. The integral on the right hand side is simpler than our original integral, but to evaluate it we need to again use integration by parts.

To find $\int \theta \sin 3\theta \, d\theta$, let $u = \theta$ and $v' = \sin 3\theta$, so $u' = 1$ and $v = -\frac{1}{3}\cos 3\theta$.

This gives

$$\int \theta \sin 3\theta \, d\theta = -\frac{1}{3}\theta \cos 3\theta + \frac{1}{3}\int \cos 3\theta \, d\theta = -\frac{1}{3}\theta \cos 3\theta + \frac{1}{9}\sin 3\theta + C.$$

Thus,

$$\int \theta^2 \cos 3\theta d\theta = \frac{1}{3}\theta^2 \sin 3\theta + \frac{2}{9}\theta \cos 3\theta - \frac{2}{27}\sin 3\theta + C.$$

32. $\displaystyle\int_3^5 x\cos x\,dx = (\cos x + x\sin x)\Big|_3^5 = \cos 5 + 5\sin 5 - \cos 3 - 3\sin 3 \approx -3.944.$

33. We use integration by parts. Let $u = z$ and $v' = e^{-z}$, so $u' = 1$ and $v = -e^{-z}$. Then

$$\begin{aligned}\int_0^{10} ze^{-z}\,dz &= -ze^{-z}\Big|_0^{10} + \int_0^{10} e^{-z}\,dz\\ &= -10e^{-10} + (-e^{-z})\Big|_0^{10}\\ &= -11e^{-10} + 1\\ &\approx 0.9995.\end{aligned}$$

34. $\displaystyle\int_1^3 t\ln t\,dt = \left(\frac{1}{2}t^2\ln t - \frac{1}{2}t\right)\Big|_1^3 = \frac{9}{2}\ln 3 - 2 \approx 2.944.$

35. We use integration by parts. Let $u = \arctan y$ and $v' = 1$, so $u' = \frac{1}{1+y^2}$ and $v = y$. Thus

$$\begin{aligned}\int_0^1 \arctan y\,dy &= (\arctan y)y\Big|_0^1 - \int_0^1 \frac{y}{1+y^2}\,dy\\ &= \frac{\pi}{4} - \frac{1}{2}\ln|1+y^2|\Big|_0^1\\ &= \frac{\pi}{4} - \frac{1}{2}\ln 2 \approx 0.439.\end{aligned}$$

36. $\displaystyle\int_0^5 \ln(1+t)\,dt = ((1+t)\ln(1+t) - (1+t))\Big|_0^5 = 6\ln 6 - 5 \approx 5.751.$

37. We use integration by parts. Let $u = \arcsin z$ and $v' = 1$, so $u' = \dfrac{1}{\cdots}$ and $v = z$. Then

48. Integration by parts: let $u = \cos\theta$ and $v' = \cos\theta$, so $u' = -\sin\theta$ and $v = \sin\theta$.

$$\int \cos^2\theta\, d\theta = \sin\theta\cos\theta - \int (-\sin\theta)(\sin\theta)\, d\theta$$

$$= \sin\theta\cos\theta + \int \sin^2\theta\, d\theta.$$

Now use $\sin^2\theta = 1 - \cos^2\theta$.

$$\begin{aligned}\int \cos^2\theta\, d\theta &= \sin\theta\cos\theta + \int (1-\cos^2\theta)\, d\theta \\ &= \sin\theta\cos\theta + \int d\theta - \int \cos^2\theta\, d\theta.\end{aligned}$$

Adding $\int \cos^2\theta\, d\theta$ to both sides, we have

$$\begin{aligned}2\int \cos^2\theta\, d\theta &= \sin\theta\cos\theta + \theta + C \\ \int \cos^2\theta\, d\theta &= \frac{1}{2}\sin\theta\cos\theta + \frac{1}{2}\theta + C'.\end{aligned}$$

Use the identity $\cos^2\theta = \frac{1+\cos 2\theta}{2}$.

$$\int \cos^2\theta\, d\theta = \int \frac{1+\cos 2\theta}{2}\, d\theta = \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta + C.$$

The only difference is in the two terms $\frac{1}{2}\sin\theta\cos\theta$ and $\frac{1}{4}\sin 2\theta$, but since $\sin 2\theta = 2\sin\theta\cos\theta$, we have $\frac{1}{4}\sin 2\theta = \frac{1}{4}(2\sin\theta\cos\theta) = \frac{1}{2}\sin\theta\cos\theta$, so there is no real difference between the formulas.

49. First, let $u = e^x$ and $v' = \sin x$, so $u' = e^x$ and $v = -\cos x$.
Thus $\int e^x \sin x\, dx = -e^x\cos x + \int e^x \cos x\, dx$. To calculate $\int e^x\cos x\, dx$, we again need to use integration by parts. Let $u = e^x$ and $v' = \cos x$, so $u' = e^x$ and $v = \sin x$.
Thus

$$\int e^x \cos x\, dx = e^x\sin x - \int e^x \sin x\, dx.$$

This gives

$$\int e^x \sin x\, dx = e^x\sin x - e^x\cos x - \int e^x \sin x\, dx.$$

By adding $\int e^x \sin x\, dx$ to both sides, we obtain

$$2\int e^x \sin x\, dx = e^x(\sin x - \cos x) + C.$$

$$\text{Thus } \int e^x \sin x\, dx = \frac{1}{2}e^x(\sin x - \cos x) + C.$$

This problem could also be done in other ways; for example, we could have started with $u = \sin x$ and $v' = e^x$ as well.

50. Let $u = e^\theta$ and $v' = \cos\theta$, so $u' = e^\theta$ and $v = \sin\theta$. Then $\int e^\theta\cos\theta\, d\theta = e^\theta\sin\theta - \int e^\theta \sin\theta\, d\theta$.
In Problem 49 we found that $\displaystyle\int e^x \sin x dx = \frac{1}{2}e^x(\sin x - \cos x) + C$.

$$\begin{aligned}\int e^\theta\cos\theta\, d\theta &= e^\theta\sin\theta - \left[\frac{1}{2}e^\theta(\sin\theta - \cos\theta)\right] + C \\ &= \frac{1}{2}e^\theta(\sin\theta + \cos\theta) + C.\end{aligned}$$

51. We int

58. Since $f'(x) = 2x$, integration by parts tells us that

$$\int_0^{10} f(x)g'(x)\,dx = f(x)g(x)\Big|_0^{10} - \int_0^{10} f'(x)g(x)\,dx$$
$$= f(10)g(10) - f(0)g(0) - 2\int_0^{10} xg(x)\,dx.$$

We can use left and right Riemann Sums with $\Delta x = 2$ to approximate $\int_0^{10} xg(x)\,dx$:

$$\text{Left sum} \approx 0\cdot g(0)\Delta x + 2\cdot g(2)\Delta x + 4\cdot g(4)\Delta x + 6\cdot g(6)\Delta x + 8\cdot g(8)\Delta x$$
$$= (0(2.3) + 2(3.1) + 4(4.1) + 6(5.5) + 8(5.9))\,2 = 205.6.$$
$$\text{Right sum} \approx 2\cdot g(2)\Delta x + 4\cdot g(4)\Delta x + 6\cdot g(6)\Delta x + 8\cdot g(8)\Delta x + 10\cdot g(10)\Delta x$$
$$= (2(3.1) + 4(4.1) + 6(5.5) + 8(5.9) + 10(6.1))\,2 = 327.6.$$

A good estimate for the integral is the average of the left and right sums, so

$$\int_0^{10} xg(x)\,dx \approx \frac{205.6 + 327.6}{2} = 266.6.$$

Substituting values for f and g, we have

$$\int_0^{10} f(x)g'(x)\,dx = f(10)g(10) - f(0)g(0) - 2\int_0^{10} xg(x)\,dx$$
$$\approx 10^2(6.1) - 0^2(2.3) - 2(266.6) = 76.8 \approx 77.$$

59.