2. With $r = \sqrt{3}$ and $\theta = -3\pi/4$, we find $x = r\cos\theta = \sqrt{3}\cos(-3\pi/4) = \sqrt{3}(-\sqrt{2}/2) = -\sqrt{6}/2$ and $y = \sqrt{3}\sin(-3\pi/4) = \sqrt{3}(-\sqrt{2}/2) = -\sqrt{6}/2$.

The rectangular coordinates are $(-\sqrt{6}/2, -\sqrt{6}/2)$.

...rrect θ. The polar

8. With $x = -\sqrt{3}$ and $y = 1$, find $r = \sqrt{(-\sqrt{3})^2 + 1^2} = \sqrt{4} = 2$. Find θ from $\tan\theta = y/x = 1/(-\sqrt{3})$. Thus, $\theta = \tan^{-1}(-1/\sqrt{3}) = -\pi/6$. Since $(-\sqrt{3}, 1)$ is in the second quadrant, $\theta = -\pi/6 + \pi = 5\pi/6$. The polar coordinates are $(2, 5\pi/6)$.

9. (a) Table 8.5 ...

16. (a) Let $0 \leq \theta \leq \pi/4$ and $0 \leq r \leq 1$.
(b) Break the region into two pieces: one with $0 \leq x \leq \sqrt{2}/2$ and $0 \leq y \leq x$, the other with $\sqrt{2}/2 \leq x \leq 1$ and $0 \leq y \leq \sqrt{1-x^2}$.

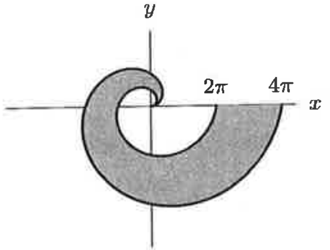

Figure 8.51: Region between the inner spiral, $r = \theta$, and the outer spiral, $r = 2\theta$

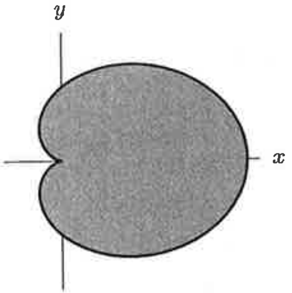

Figure 8.52: Cardioid $r = 1 + \cos\theta$

26. The cardioid is shown in Figure 8.52. The following integral can be evaluated using a calculator or by parts or usin table of integrals.

$$\begin{aligned}\text{Area } &= \frac{1}{2}\int_0^{2\pi} (1+\cos\theta)^2\, d\theta = \frac{1}{2}\int_0^{2\pi} \left(1 + 2\cos\theta + \cos^2\theta\right)\, d\theta \\ &= \frac{1}{2}\left(\theta + 2\sin\theta + \frac{1}{2}\cos\theta\sin\theta + \frac{1}{2}\theta\right)\Big|_0^{2\pi} = \frac{1}{2}(2\pi + 0 + 0 + \pi) = \frac{3\pi}{2}.\end{aligned}$$

42. Let $f(\theta) = \theta$. Then $f'(\theta) = 1$. Using the formula derived in Problem 41, we have

$$\text{Arclength} = \int_0^{2\pi} \sqrt{1+\theta^2}\, d\theta = \left[\frac{1}{2}\theta\sqrt{1+\theta^2} + \frac{1}{2}\ln\left|\theta + \sqrt{1+\theta^2}\right|\right]_0^{\pi} = \frac{1}{2}\pi\sqrt{1+\pi^2} + \frac{1}{2}\ln\left|\pi + \sqrt{1+\pi^2}\right|$$

43. Let $f(\theta) = 1 + \cos\theta$. Then $f'(\theta) = -\sin\theta$. Using numerical methods, we have

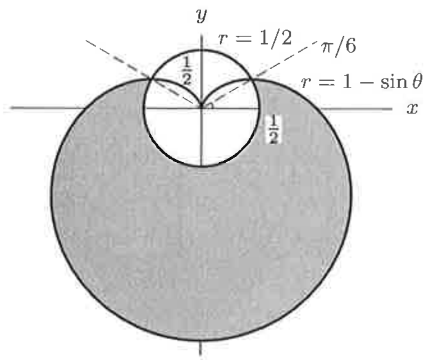

Figure 8.55

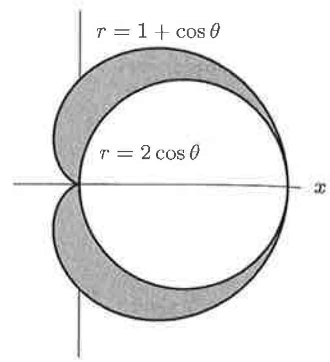

Figure 8.56

30. Figure 8.56 shows the curves which touch at $(2, 0)$ and the origin. However, the circle lies entirely inside the cardioid, so we find the area by subtracting the area of the circle from that of the cardioid. To find the areas, we take the integrals.

The cardioid, $r = 1 + \cos\theta$, starts at $(2, 0)$ when $\theta = 0$ and traces the top half, reaching the origin when $\theta = \pi$. Thus

$$\text{Area of cardioid } = 2 \cdot \frac{1}{2} \int_0^{\pi} (1 + \cos\theta)^2 \, d\theta.$$

The circle starts at $(2, 0)$ when $\theta = 0$ and traces the top half, reaching the origin when $\theta = \pi/2$. Thus

$$\text{Area of circle } = 2 \cdot \frac{1}{2} \int_0^{\pi/2} (2\cos\theta)^2 \, d\theta.$$

The area, A, we want is therefore

$$\begin{aligned}
\text{Area } &= 2 \cdot \frac{1}{2} \int_0^{\pi} (1 + \cos\theta)^2 \, d\theta - 2 \cdot \frac{1}{2} \int_0^{\pi/2} (2\cos\theta)^2 \, d\theta \\
&= \int_0^{\pi} (1 + 2\cos\theta + \cos^2\theta) \, d\theta - \int_0^{\pi/2} 4\cos^2\theta \, d\theta \\
&= \left(\theta + 2\sin\theta + \frac{1}{2}(\sin\theta\cos\theta + \theta)\right)\Big|_0^{\pi} - \frac{4}{2}(\sin\theta\cos\theta + \theta)\Big|_0^{\pi/2} \\
&= \frac{3}{2}\pi - 2 \cdot \frac{\pi}{2} = \frac{\pi}{2}.
\end{aligned}$$

Alternatively, we could compute the area of the cardioid and subtract the area of the circle of radius 1 from it.

The integrals can be computed numerically using a calculator, or, as we show, using integration by parts or formula IV-18 from the integral tables.

31. (a) The graph of $r = 2\cos\theta$ is a circle of radius 1 centered at $(1, 0)$; the graph of $r = 2\sin\theta$ is a circle of radius 1 centered at $(0, 1)$. See Figure 8.57.

(b) The Cartesian coordinates of the points of intersection are at $(0, 0)$ and $(1, 1)$.

The origin corresponds to $\theta = \pi/2$ on $r = 2\cos\theta$ and to $\theta = 0$ on $r = 2\sin\theta$. The point $(1, 1)$ has polar coordinates $r = \sqrt{2}, \theta = \pi/4$.

We find the area below the line $\theta = \pi/4$ and above $r = 2\sin\theta$ and double it:

$$\text{Area } = 2 \cdot \frac{1}{2} \int_0^{\pi/4} (2\sin\theta)^2 \, d\theta = 4 \int_0^{\pi/4} \sin^2\theta \, d\theta.$$

Using a calculator, integration by parts or formula IV-17 from the integral tables,

$$\text{Area } = 4\left(-\frac{1}{2}\sin\theta\cos\theta + \frac{\theta}{2}\right)\Big|_0^{\pi/4} = -2 \cdot \frac{1}{2} + 2\frac{\pi}{4} = \frac{\pi}{2} - 1.$$

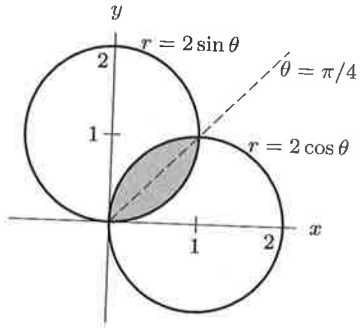

Figure 8.57

32. The area is

$$A = \frac{1}{2}\int_0^a r^2\, d\theta = \frac{1}{2}\int_0^a \theta^2\, d\theta = 1$$

$$\frac{1}{2}\left(\frac{\theta^3}{3}\right)\bigg|_0^a = 1$$

$$\frac{a^3}{6} = 1$$

$$a^3 = 6$$

$$a = \sqrt[3]{6}.$$

33. (a) See Figure 8.58.
(b) The curves intersect when $r^2 = 2$

$$4\cos 2\theta = 2$$

$$\cos 2\theta = \frac{1}{2}.$$

In the first quadrant:

$$2\theta = \frac{\pi}{3} \quad \text{so} \quad \theta = \frac{\pi}{6}.$$

Using symmetry, the area in the first quadrant can be multiplied by 4 to find the area of the total bounded region.

$$\text{Area} = 4\left(\frac{1}{2}\right)\int_0^{\pi/6} (4\cos 2\theta - 2)\, d\theta$$

$$= 2\left(\frac{4\sin 2\theta}{2} - 2\theta\right)\bigg|_0^{\pi/6}$$

$$= 4\sin\frac{\pi}{3} - \frac{2}{3}\pi$$

$$= 4\frac{\sqrt{3}}{2} - \frac{2}{3}\pi$$

$$= 2\sqrt{3} - \frac{2}{3}\pi = 1.370.$$

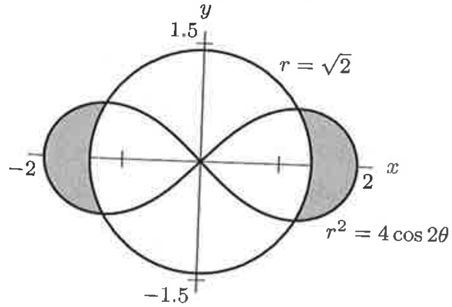

Figure 8.58

37. (a) Expressing x and y parametrically in terms of θ, we have

$$x = r\cos\theta = \frac{\cos\theta}{\theta} \quad \text{and} \quad y = r\sin\theta = \frac{\sin\theta}{\theta}.$$

The slope of the tangent line is given by

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \left(\frac{\theta\cos\theta - \sin\theta}{\theta^2}\right) \Big/ \left(\frac{-\theta\sin\theta - \cos\theta}{\theta^2}\right) = \frac{\sin\theta - \theta\cos\theta}{\cos\theta + \theta\sin\theta}.$$

At $\theta = \pi/2$, we have

$$\left.\frac{dy}{dx}\right|_{\theta=\pi/2} = \frac{1 - (\pi/2)0}{0 + (\pi/2)1} = \frac{2}{\pi}.$$

At $\theta = \pi/2$, we have $x = 0$, $y = 2/\pi$, so the equation of the tangent line is

$$y = \frac{2}{\pi}x + \frac{2}{\pi}.$$

(b) As $\theta \to 0$,

$$x = \frac{\cos\theta}{\theta} \to \infty \quad \text{and} \quad y = \frac{\sin\theta}{\theta} \to 1.$$

Thus, $y = 1$ is a horizontal asymptote. See Figure 8.59.

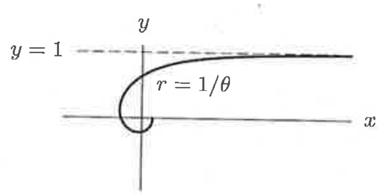

Figure 8.59

38. The limaçon is given by $r = 1 + 2\cos\theta$; see Figure 8.60. At $\theta = 0$, the graph is at $(3, 0)$; as θ increases, the graph sweeps out the top arc (on which the maximum value of y occurs), reaching the origin when

$$\begin{aligned} 1 + 2\cos\theta &= 0 \\ \cos\theta &= -\frac{1}{2} \\ \theta &= \frac{2\pi}{3}. \end{aligned}$$

Thus, we want to find the maximum value of y on the interval $0 \le \theta \le 2\pi/3$. Since $y = r\sin\theta$, we want to find the maximum value of

$$y = (1 + 2\cos\theta)\sin\theta = \sin\theta + 2\cos\theta\sin\theta.$$

At a critical point

$$\begin{aligned} \frac{dy}{d\theta} = \cos\theta - 2\sin^2\theta + 2\cos^2\theta &= 0 \\ \cos\theta - 2(1 - \cos^2\theta) + 2\cos^2\theta &= 0 \\ 4\cos^2\theta + \cos\theta - 2 &= 0 \\ \cos\theta &= \frac{-1 \pm \sqrt{33}}{8} = 0.593, -0.843. \end{aligned}$$

Thus, $\theta = \cos^{-1}(0.593) = 0.936$ and $\theta = \cos^{-1}(-0.843) = 2.574$ are the critical values. Since 2.574 is outside the interval $0 \le \theta \le 2\pi/3$, there is one critical point $\theta = 0.963$.

At the endpoints of the interval, $y = 0$. At $\theta = 0.936$, we have $y = 1.760$, which is the maximum value.

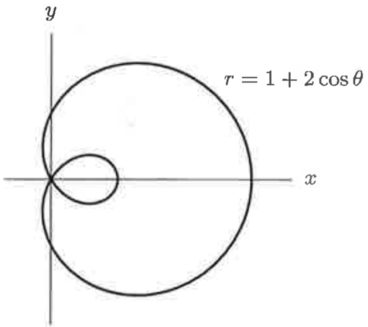

Figure 8.60: The inner loop has $r < 0$

39. Since $x = \theta\cos\theta$ and $y = \theta\sin\theta$, we have

$$\text{Arc length } = \int_0^{2\pi} \sqrt{(\cos\theta - \theta\sin\theta)^2 + (\sin\theta + \theta\cos\theta)^2}\, d\theta = \int_0^{2\pi} \sqrt{1+\theta^2}\, d\theta = 21.256.$$