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Solutions for Section 9.3

Exercises

1. The series is $1+2+3+4+5+\cdots$. The sequence of partial sums is

$$S_1=1,\quad S_2=1+2,\quad S_3=1+2+3,\quad S_4=1+2+3+4,\quad S_5=1+2+3+4+5,\ldots$$

which is

$$1,\quad 3,\quad 6,\quad 10,\quad 15\ldots.$$

2. The series is $-1+1/2-1/3+1/4-1/5+\cdots$. The sequence of partial sums is

$$S_1=-1,\quad S_2=-1+\frac{1}{2},\quad S_3=-1+\frac{1}{2}-\frac{1}{3},\quad S_4=-1+\frac{1}{2}-\frac{1}{3}+\frac{1}{4},\quad S_5=-1+\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5},\ldots$$

which is

$$-1,\quad -\frac{1}{2},\quad -\frac{5}{6},\quad -\frac{7}{12},\quad -\frac{47}{60},\ldots.$$

$n=1$

16. Since the terms in the series are positive and decreasing, we can use the integral test. We calculate the co
improper integral using the substitution $w = x^2$

$$\int_0^{\infty} \frac{2x}{1+x^4}\,dx = \lim_{b\to\infty} \int_0^{b} \frac{2x}{1+x^4}\,dx = \lim_{b\to\infty} \arctan x^2 \Big|_0^b = \lim_{b\to\infty} \arctan b^2 = \frac{\pi}{2}.$$

Since the limit exists, the integral converges, so the series $\displaystyle\sum_{n=0}^{\infty} \frac{2n}{1+n^4}$ converges.

... the integral test. We calculate the c ostitution

28. Let $a_n = (\ln n)/n$ and $f(x) = (\ln x)/x$. We use the integral test and consider the improper integral

$$\int_c^\infty \frac{\ln x}{x}\,dx.$$

Since

$$\int_c^R \frac{\ln x}{x}\,dx = \frac{1}{2}(\ln x)^2\Big|_c^R = \frac{1}{2}\left((\ln R)^2 - (\ln c)^2\right),$$

and $\ln R$ grows without bound as $R \to \infty$, the integral diverges. Therefore, the integral test tells us that the series, $\sum_{n=1}^{\infty} \frac{\ln n}{n}$, also diverges.

48. (a) The left-hand sum approximation to $\int_1^n \frac{1}{x}\,dx$ in Figure 9.6 shows that

$$\begin{aligned}
\int_1^n \frac{dx}{x} &< 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \\
\ln n - \ln 1 &< 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \\
0 &< 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n.
\end{aligned}$$

Thus, $0 < a_n$.

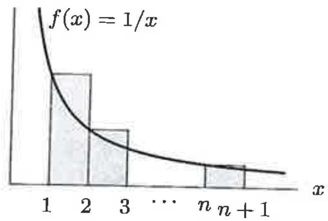

Figure 9.6

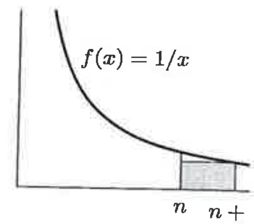

Figure 9.7

(b) We calculate

$$\begin{aligned}
a_n - a_{n+1} &= \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right) - \ln n - \left(\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n+1}\right) - \ln(n \cdot\right. \\
&= \ln(n+1) - \ln n - \frac{1}{n+1}.
\end{aligned}$$

But using the right sum with one rectangle in Figure 9.7, we see that

$$\begin{aligned}
\int_n^{n+1} \frac{dx}{x} &> \frac{1}{n+1} \\
\ln(n+1) - \ln n &> \frac{1}{n+1}.
\end{aligned}$$

Thus

$$\begin{aligned}
a_n - a_{n+1} = \ln(n+1) - \ln n - \frac{1}{n+1} &> 0. \\
a_n &> a_{n+1}.
\end{aligned}$$

(c) Since a_n is a decreasing sequence bounded below by 0, Theorem 9.1 ensures that $\lim_{n\to\infty} a_n$ exists.

(d) The sequence converges slowly, but a calculator or computer gives $a_{200} = 0.5797$. For comparison, a $a_{500} = 0.5782$. Thus, $\gamma = 0.58$. More extensive calculations show that $\gamma = 0.577216$.

49. (a) A calculator or computer gives

10. (a) Suppose $q_i(x)$ is the quadratic function approximating $f(x)$ on the subinterval $[x_i, x_{i+1}]$, and m_i is the midpoint of the interval, $m_i = (x_i + x_{i+1})/2$. Then, using the equation in Problem 9, with $a = x_i$ and $b = x_{i+1}$ and $h = \Delta x = x_{i+1} - x_i$:

$$\int_{x_i}^{x_{i+1}} f(x)dx \approx \int_{x_i}^{x_{i+1}} q_i(x)dx = \frac{\Delta x}{3}\left(\frac{q_i(x_i)}{2} + 2q_i(m_i) + \frac{q_i(x_{i+1})}{2}\right).$$

(b) Summing over all subintervals gives

$$\int_a^b f(x)dx \approx \sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}} q_i(x)dx = \sum_{i=0}^{n-1}\frac{\Delta x}{3}\left(\frac{q_i(x_i)}{2} + 2q_i(m_i) + \frac{q_i(x_{i+1})}{2}\right).$$

Splitting the sum into two parts:

$$= \frac{2}{3}\sum_{i=0}^{n-1} q_i(m_i)\Delta x + \frac{1}{3}\sum_{i=0}^{n-1}\frac{q_i(x_i) + q_i(x_{i+1})}{2}\Delta x$$
$$= \frac{2}{3}\mathrm{MID}(n) + \frac{1}{3}\mathrm{TRAP}(n)$$
$$= \mathrm{SIMP}(n).$$

Solutions for Section 7.7

Exercises

1. (a) See Figure 7.30. The area extends out infinitely far along the positive x-axis.

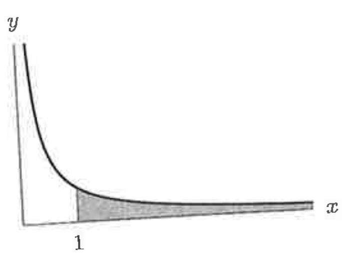

Figure 7.30

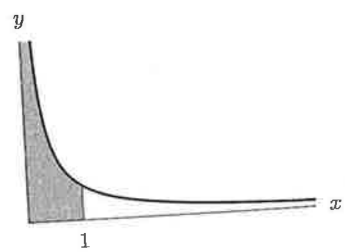

Figure 7.31

(b) See Figure 7.31. The area extends up infinitely far along the positive y-axis.

2. We have

$$\int_0^\infty e^{-0.4x}dx = \lim_{b\to\infty}\int_0^b e^{-0.4x}dx = \lim_{b\to\infty}(-2.5e^{-0.4x})\big|_0^b = \lim_{b\to\infty}(-2.5e^{-0.4b} + 2.5).$$

As $b \to \infty$, we know $e^{-0.4b} \to 0$ and so we see that the integral converges to 2.5. See Figure 7.32. The area conti[nues] indefinitely out to the right.

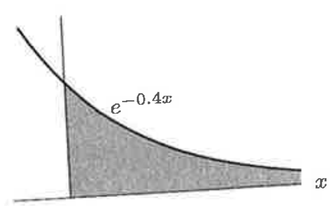

Figure 7.32

8. Use the substitution $w = \sqrt{x}$. Since $dw = \frac{1}{2}x^{-1/2}\,dx$, we have $dx = 2w\,dw$, so

$$\int e^{-\sqrt{x}}\,dx = 2\int we^{-w}\,dw.$$

Note that this substitution leaves the limits unchanged. Using integration by parts with $u = w$ and $v' = e^{-w}$, we fi

$$\begin{aligned} 2\int we^{-w}\,dw &= 2\left[-we^{-w} + \int e^{-w}\,dw\right] \\ &= 2[-we^{-w} - e^{-w}]. \end{aligned}$$

So,

$$\begin{aligned} \int_0^\infty we^{-w}\,dx &= \lim_{b\to\infty} 2\left[-we^{-w} - e^{-w}\right]_0^b \\ &= 2\lim_{b\to\infty}(-be^{-b} - e^{-b} + 1) \\ &= 2\left[\lim_{b\to\infty} -\frac{b+1}{e^b} + 1\right] \\ &= 2\left[\lim_{b\to\infty} -\frac{1}{e^b} + 1\right] \quad \text{by l'Hopital} \\ &= 2. \end{aligned}$$

9. We h

which diverges. Therefore the area is infinite.

40. The factor $\ln x$ grows slowly enough not to change the convergence or divergence of the integral, although it wi what it converges or diverges to.

Integrating by parts or using the table of integrals, we get

$$\begin{aligned}\int_e^\infty x^p \ln x\, dx &= \lim_{b\to\infty}\int_e^b x^p \ln x\, dx \\ &= \lim_{b\to\infty}\left[\frac{1}{p+1}x^{p+1}\ln x - \frac{1}{(p+1)^2}x^{p+1}\right]\Bigg|_e^b \\ &= \lim_{b\to\infty}\left[\left(\frac{1}{p+1}b^{p+1}\ln b - \frac{1}{(p+1)^2}b^{p+1}\right)\right. \\ &\qquad \left. - \left(\frac{1}{p+1}e^{p+1} - \frac{1}{(p+1)^2}e^{p+1}\right)\right].\end{aligned}$$

If $p > -1$, then $(p+1)$ is positive and the limit does not exist since b^{p+1} and $\ln b$ both approach ∞ as b doe

If $p < -1$, then $(p+1)$ is negative and both b^{p+1} and $b^{p+1}\ln b$ approach 0 as $b \to \infty$. (This follows by l at graphs of $x^{p+1}\ln x$ (for different values of p), or by noting that $\ln x$ grows more slowly than x^{p+1} tends to 0.) value of the integral is $-pe^{p+1}/(p+1)^2$.

The case $p = -1$ has to be handled separately. For $p = -1$,

$$\int_e^\infty \frac{\ln x}{x}\, dx = \lim_{b\to\infty}\int_e^b \frac{\ln x}{x}\, dx = \lim_{b\to\infty}\frac{(\ln x)^2}{2}\Bigg|_e^b = \lim_{b\to\infty}\left(\frac{(\ln b)^2 - 1}{2}\right).$$

As $b \to \infty$, this limit does not exist, so the integral diverges if $p = -1$.

To summarize, $\int_e^\infty x^p \ln x\, dx$ converges for $p < -1$ to the value $-pe^{p+1}/(p+1)^2$.

41. The factor $\ln x$ grows slowly enough (as $x \to 0^+$) not to change the convergence or divergence of the

2. Let $a_n = 1/(n^2+2)$. Since $n^2+2 > n^2$, we have $1/(n^2+2) < 1/n^2$, so

$$0 < a_n < \frac{1}{n^2}.$$

The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, so the comparison test tells us that the series $\sum_{n=1}^{\infty} \frac{1}{n^2+2}$ also converges.

36. We have

$$\frac{a_n}{b_n} = \frac{((1+n)/(3n))^n}{(1/3)^n} = \left(\frac{n+1}{n}\right)^n = \left(1+\frac{1}{n}\right)^n,$$

so

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e = c \neq 0.$$

Since $\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$ is a convergent geometric series, the original series converges.

also diverges.

40. The n^{th} term $a_n = (n^3 - 2n^2 + n + 1)/(n^4 - 2)$ behaves like $n^3/n^4 = 1/n$ for large n, so we take $b_n = 1/n$. We have

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{(n^3 - 2n^2 + n + 1)/(n^4 - 2)}{1/n} = \lim_{n\to\infty} \frac{n^4 - 2n^3 + n^2 + n}{n^4 - 2} = 1.$$

The limit comparison test applies with $c = 1$. The harmonic series $\sum 1/n$ diverges. Thus $\sum \left(n^3 - 2n^2 + n + 1\right) / \left(n^4 - 2\right)$ also diverges.

41. The n^{th} term $a_n = 2^n/(3^n - 1)$ behaves like $2^n/3^n$ for large n, so we take $b_n = 2^n/3^n$. We have

58. The first few terms of the series may be written

$$1 + e^{-1} + e^{-2} + e^{-3} + \cdots;$$

this is a geometric series with $a = 1$ and $x = e^{-1} = 1/e$. Since $|x| < 1$, the geometric series converg

$$S = \frac{1}{1-x} = \frac{1}{1-e^{-1}} = \frac{e}{e-1}.$$