14. Since $a_n = 1/(2n)!$, replacing n by $n+1$ gives $a_{n+1} = 1/(2n+2)!$. Thus

$$\frac{|a_{n+1}|}{|a_n|} = \frac{\frac{1}{(2n+2)!}}{\frac{1}{(2n)!}} = \frac{(2n)!}{(2n+2)!} = \frac{(2n)!}{(2n+2)(2n+1)(2n)!} = \frac{1}{(2n+2)(2n+1)},$$

so

$$L = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{1}{(2n+2)(2n+1)} = 0.$$

Since $L = 0$, the ratio test tells us that $\sum_{n=1}^{\infty} \frac{1}{(2n)!}$ converges.

15. Since $a_n = (n!)^2/(2n)!$, replacin...

22. Since $(-1)^n \cos(n\pi) = (-1)^{2n} = 1$, this is not an alternating series.

23. Since $a_n = \cos n$ is not always positive, this is not an alternating series.

24. Let $a_n = 1/\sqrt{n}$. Then replacing n by $n+1$ we have $a_{n+1} = 1/\sqrt{n+1}$. Since $\sqrt{n+1} > \sqrt{n}$, we have $\frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}}$, hence $a_{n+1} < a_n$. In addition, $\lim_{n\to\infty} a_n = 0$ so $\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converges by the alternating series test.

25. Let $a_n = 1/(2n+1)$. Then replacing n by $n+1$ gives $a_{n+1} = 1/(2n+3)$. Since $2n+3 > 2n+1$, we have

$$0 < a_{n+1} = \frac{1}{2n+3} < \frac{1}{2n+1} = a_n.$$

We also have $\lim_{n\to\infty} a_n = 0$. Therefore, the alternating series test tells us that the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n+1}$ converges.

26. Let $a_n = 1/(n^2+2n+1) = 1/(n+1)^2$. Then replacing n by $n+1$ gives $a_{n+1} = 1/(n+2)^2$. Since $n+2 > n+1$, we have

$$\frac{1}{(n+2)^2} < \frac{1}{(n+1)^2}$$

so

$$0 < a_{n+1} < a_n.$$

We also have $\lim_{n\to\infty} a_n = 0$. Therefore, the alternating series test tells us that the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2+2n+1}$ converges.

27. Let $a_n = 1/e^n$. Then replacing n by $n+1$ we have $a_{n+1} = 1/e^{n+1}$. Since $e^{n+1} > e^n$, we have $\frac{1}{e^{n+1}} < \frac{1}{e^n}$, hence $a_{n+1} < a_n$. In addition, $\lim_{n\to\infty} a_n = 0$ so $\sum_{n=1}^{\infty} \frac{(-1)^n}{e^n}$ converges by the alternating series test. We can also observe that the series is geometric with ratio $x = -1/e$ can hence converges since $|x| < 1$.

28. Both $\sum \frac{(-1)^n}{2^n} = \sum \left(\frac{-1}{2}\right)^n$ and $\sum \frac{1}{2^n} = \sum \left(\frac{1}{2}\right)^n$ are convergent geometric series. Thus $\sum \frac{(-1)^n}{2^n}$ is absolutely convergent.

29. The series $\sum \frac{(-1)^n}{2n}$ converges by the alternating series test. However $\sum \frac{1}{2n}$ diverges because it is a multiple of the harmonic series. Thus $\sum \frac{(-1)^n}{2n}$ is conditionally convergent.

30. Since

$$\lim_{n\to\infty} \left(1 + \frac{1}{n^2}\right) = 1,$$

the n^{th} term $a_n = (-1)^n \left(1 + \frac{1}{n^2}\right)$ does not tend to zero as $n \to \infty$. Thus, the series $\sum (-1)^n \left(1 + \frac{1}{n^2}\right)$ is divergent.

Solutions for Section 9.5

Exercises

1. Yes.
2. No, because it contains negative powers of x.
3. No, each term is a power of a different quantity.
4. Yes. It's a polynomial, or a series with all coefficients beyond the 7th being zero.
5. The general term can be written as $\frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n \cdot n!}x^n$ for $n \geq 1$. Other answers are possible.
6. The general term can be written as $\frac{p(p-1)(p-2)\cdots(p-n+1)}{n!}x^n$ for $n \geq 1$. Other answers are possible.
7. The general term can be written as $\frac{(-1)^k(x-1)^{2k}}{(2k)!}$ for $k \geq 0$. Other answers are possible.
8. The general term can be written as $\frac{(-1)^{k+1}(x-1)^{2k+1}}{(2(k-1))!}$ for $k \geq 1$ or as $\frac{(-1)^k(x-1)^{2k+3}}{(2k)!}$ for $k \geq 0$. Other answers are possible.

12. Since $C_n = n^3$, replacing n by $n+1$ gives $C_{n+1} = (n+1)^3$. Using the ratio test, with $a_n = n^3 x^n$, we have

$$\frac{|a_{n+1}|}{|a_n|} = |x|\frac{|C_{n+1}|}{|C_n|} = |x|\frac{(n+1)^3}{n^3} = |x|\left(\frac{n+1}{n}\right)^3.$$

We have

$$\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|} = |x|.$$

Thus the radius of convergence is $R = 1$.

13. Since $C_n = (n+1)/(2^n + n)$, replacing n by $n+1$ gives $C_{n+1} = (n+2)/(2^{n+1} + n + 1)$. Using the ratio

18. Here the coefficient of the n^{th} term is $C_n = (2^n/n!)$. Now we have

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(2^{n+1}/(n+1)!)x^{n+1}}{(2^n/n!)x^n}\right| = \frac{2|x|}{n+1} \to 0 \text{ as } n \to \infty.$$

Thus, the radius of convergence is $R = \infty$, and the series converges for all x.

19. Here $C_n = (2n)!/(n!)^2$. We have:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(2(n+1))!/((n+1)!)^2 x^{n+1}}{(2n)!/(n!)^2 x^n}\right| = \frac{(2(n+1))!}{(2n)!} \cdot \frac{(n!)^2}{((n+1)!)^2}|x|$$
$$= \frac{(2n+2)(2n+1)|x|}{(n+1)^2} \to 4|x| \text{ as } n \to \infty.$$

Thus, the radius of convergence is $R = 1/4$.

20. Here the coefficient of the n^{th} term is $C_n = (2n+1)/n$. Applying the ratio test, we consider:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{((2n+3)/(n+1))x^{n+1}}{((2n+1)/n)x^n}\right| = |x|\frac{2n+3}{2n+1} \cdot \frac{n}{n+1} \to |x| \text{ as } n \to \infty.$$

Thus, the radius of convergence is $R = 1$.

21. We write the series as

$$x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + (-1)^{n-1}\frac{x^{2n-1}}{2n-1} + \cdots,$$

so

$$a_n = (-1)^{n-1}\frac{x^{2n-1}}{2n-1}.$$

Replacing n by $n+1$, we have

$$a_{n+1} = (-1)^{n+1-1}\frac{x^{2(n+1)-1}}{2(n+1)-1} = (-1)^n\frac{x^{2n+1}}{2n+1}.$$

Thus

$$\frac{|a_{n+1}|}{|a_n|} = \left|\frac{(-1)^n x^{2n+1}}{2n+1}\right| \cdot \left|\frac{2n-1}{(-1)^{n-1}x^{2n-1}}\right| = \frac{2n-1}{2n+1}x^2,$$

so

$$L = \lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty}\frac{2n-1}{2n+1}x^2 = x^2.$$

By the ratio test, this series converges if $L < 1$, that is, if $x^2 < 1$, so $R = 1$.

Problems

22. (a) The general term of the series is x^n/n if n is odd and $-x^n/n$ if n is even, so $C_n = (-1)^{n-1}/n$, and we can use the ratio test. We have

$$\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|} = |x|\lim_{n\to\infty}\frac{|(-1)^n/(n+1)|}{|(-1)^{n-1}/n|} = |x|\lim_{n\to\infty}\frac{n}{n+1} = |x|.$$

Therefore the radius of convergence is $R = 1$. This tells us that the power series converges for $|x| < 1$ and does not converge for $|x| > 1$. Notice that the radius of convergence does not tell us what happens at the endpoints, $x = \pm 1$.

(b) The endpoints of the interval of convergence are $x = \pm 1$. At $x = 1$, we have the series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{(-1)^{n-1}}{n} + \cdots$$

This is an alternating series with $a_n = 1/n$, so by the alternating series test, it converges. At $x = -1$, we have the series

$$-1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \cdots - \frac{1}{n} - \cdots$$

This is the negative of the harmonic series, so it does not converge. Therefore the right endpoint is included, and the left endpoint is not included in the interval of convergence, which is $-1 < x \le 1$.

36. The radius of convergence, R, is between 5 and 7.

37. The series is centered at $x = -7$. Since the series converges at $x = 0$, which is a distance of 7 from $x = -7$ of convergence, R, is at least 7. Since the series diverges at $x = -17$, which is a distance of 10 from $x = -7$ of convergence is no more than 10. That is, $7 \leq R \leq 10$.

38. The radius of convergence of the series, R, is at least 4 but no larger than 7.

(a) False. Since $10 > R$ the series diverges.
(b) True. Since $3 < R$ the series converges.
(c) False. Since $1 < R$ the series converges.
(d) Not possible to determine since the radius of convergence may be more or less than 6.

39. The series is centered at $x = 3$. Since the series converges at $x = 7$, which is a distance of 4 from $x = 3$ $R \geq 4$. Since the series diverges at $x = 10$, which is a distance of 7 from $x = 3$, we know $R \leq 7$. That is, $4 \leq$

Since $x = 11$ is a distance of 8 from $x = 3$, the series diverges at $x = 11$.
Since $x = 5$ is a distance of 2 from $x = 3$, the series converges there.
Since $x = 0$ is a distance of 3 from $x = 3$, the series converges at $x = 3$.

2 points

40. (a) We use the ratio test:

$$\begin{aligned}\left|\frac{a_{n+1}}{a_n}\right| &= \left|\frac{(-1)^{n+1}x^{2(n+1)}}{2^{2(n+1)}((n+1)!)^2} \cdot \frac{2^{2n}(n!)^2}{(-1)^n x^{2n}}\right| \\ &= \frac{x^{2n+2}}{2^{2n+2}(n+1)^2(n!)^2} \cdot \frac{2^{2n}(n!)^2}{x^{2n}} \\ &= \frac{x^2}{4(n+1)^2}.\end{aligned}$$

For a fixed value of x, we have

$$\frac{x^2}{4(n+1)^2} \to 0 \quad \text{as} \quad n \to \infty.$$

The series converges for all x, so the domain of $J(x)$ is all real numbers.

(b) Since

$$J(x) = 1 - \frac{x^2}{4} + \cdots,$$

we have $J(0) = 1$.

(c) We have

$$\begin{aligned}
S_0(x) &= 1\\
S_1(x) &= 1 - \frac{x^2}{4}\\
S_2(x) &= 1 - \frac{x^2}{4} + \frac{x^4}{64}\\
S_3(x) &= 1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{2304}\\
S_4(x) &= 1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{2304} + \frac{x^8}{147{,}456}.
\end{aligned}$$

(d) The value of $J(1)$ can be approximated using partial sums. Substituting $x = 1$ into the partial sum polynomials, we have

$$\begin{aligned}
S_0(1) &= 1\\
S_1(1) &= 0.75\\
S_2(1) &= 0.765625\\
S_3(1) &= 0.765191\\
S_4(1) &= 0.765198.
\end{aligned}$$

We estimate that $J(1) \approx 0.765$. Theorem 9.9 can be used to bound the error.

(e) We see from the series that $J(x)$ is an even function, so $J(-1) = J(1)$. Thus, $J(-1) \approx 0.765$.

41. (a) We have

$$f(x) = 1 + x + \frac{x^2}{2} + \cdots,$$

so

$$f(0) = 1 + 0 + 0 + \cdots = 1.$$

(b) To find the domain of f, we find the interval of convergence.

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{|x^{n+1}/(n+1)!|}{|x^n/n!|} = \lim_{n\to\infty} \left(\frac{|x|^{n+1} n!}{|x|^n (n+1)!} \right) = |x| \lim_{n\to\infty} \frac{1}{n+1} = 0.$$

Thus the series converges for all x, so the domain of f is all real numbers.

(c) Differentiating term-by-term gives

$$\begin{aligned}
f'(x) = \frac{d}{dx}\left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right) &= \frac{d}{dx}\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right)\\
&= 0 + 1 + 2\frac{x}{2!} + 3\frac{x^2}{3!} + 4\frac{x^3}{4!} + \cdots\\
&= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots.
\end{aligned}$$

Thus, the series for f and f' are the same, so

$$f(x) = f'(x).$$

(d) We guess $f(x) = e^x$.